On the properness of the eigencurve associated to unitary Shimura curves

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ON THE PROPERNESS OF THE EIGENCURVE ASSOCIATED TO UNITARY SHIMURA CURVES

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Declaration

This thesis is a presentation of my original research work. Wherever the contributions of others are involved, every effort has been made to indicate this clearly, with due reference to the literature.

Signature: .... Date: ....

Abstract

We study overconvergent modular forms on certain unitary Shimura curves, defined for integral weights by Kassaei and general weights by Brasca. There are Hecke operators acting on these spaces of overconvergent modular forms, and there is a distinguished $U_{\mathcal{P}}$-operator which is a compact operator on these spaces.

We construct a deformation–theoretic eigencurve in this setting, which comes with a projection to the weight space. Then we prove that its nilreduction is isomorphic to the Hecke eigencurve. In particular, for each weight in the weight space, the fibre above it is identfied with systems of Hecke eigenvalues arising from overconvergent eigenforms of that weight, whose $U_{\mathcal{P}}$–eigenvalu e is not 0.

Lastly, we prove that this eigencurve is proper (that is, the map to the weight space satisfies the valuative criterion of properness). This is done using p–adic Hodge theory, via interpreting the $U_{\mathcal{P}}$–eigenvalues on the automorphic side as the eigenvalues of Frobenius on the p–adic Hodge theory side, for families of Galois representations attached to finite slope, overconvergent eigenforms.

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Contents

1. Introduction

Let p be a prime and $N \in \mathbb{N}$ be prime to p. In their paper [CM98], Coleman and Mazur defined a rigid analytic curve $\mathcal{C}_{p,N}$ called the *p-adic eigencurve of tame level* N. There is a rigid analytic group $\mathcal{W}_N$ defined over $\mathbb{Q}_p$, such that if $K/\mathbb{Q}_p$ is a finite extension, then $\mathcal{W}_N(K)$ is the group of continuous characters $\chi : \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \to K^\times$ under the usual multiplication.

There is a canonical map $\pi : \mathcal{C}_{p,N} \to \mathcal{W}_N$, which is such that if $\chi \in \mathcal{W}_N(\mathbb{C}_p)$, then there is a bijection between $\pi^{-1}(\chi) \subset \mathcal{C}_{p,N}(\mathbb{C}_p)$ and all finite-slope normalised overconvergent eigenforms of weight χ and tame level N.

One question that Coleman and Mazur posed in their original paper was:

"Do there exist p-adic analytic families of finite-slope overconvergent eigenforms, parameterised by a punctured disc, and converging at the puncture to an eigenform of *infinite* slope?"

This can be reformulated as asking, does π satisfy the *valuative criterion of properness* (or, *is the eigencurve* $\mathcal{C}_{p,N}$ *proper over the weight space*)? The requirement that the family be parameterised by a disc is important, since it was shown in [CS04] that a pointwise sequence of finite slope eigenforms can actually converge to an infinite slope newform.

In [BC06] it was proven that when $p = 2$ and $N = 1$, $\mathcal{C}_{p,N}$ is proper, while in [Cal08] Calegari refined the ideas to prove that for general p and $N \geqslant 5$, $\mathcal{C}_{p,N}$ is proper at the integral weights. It seemes as though it is difficult to generalise Buzzard and Calegari's method to other modular forms, as they use the q–expansions attached to elliptic modular forms.

In more recent work, Diao and Liu in [DL14] proved that for all p and N, $\mathcal{C}_{p,N}$ is proper. Their technique involves using the deformation–theoretic construction of the eigencurve, to convert a family of overconvergent eigenforms into a family of Galois representations (parametrised by a punctured disc). They then show that this family of Galois representations extends to the puncture.

This thesis proves that the weight map $\pi : \mathcal{C}_{p,N} \to \mathcal{W}_N$ from the eigencurve to the weight space is proper, in the sense that it satisfies the valuative criterion of properness (Theorem 6.1).

1.1. **Overview.** In chapter 2 we review some unitary Shimura curves and the modular forms associated to them. We also define the weight space, the overconvergent modular forms of general weight over these Shimura curves (this is due to Brasca, who extended Kassaei's work) and give the Hecke operators that act on these spaces of overconvergent modular forms.

In chapter 3, we briefly review the construction of the Hecke eigencurve via Buzzard's eigenvariety machine in [Buz07].

In chapter 4, we construct a deformation–theoretic eigencurve over a given unitary Shimura curve. This is analogous to the work of Coleman and Mazur, and we prove that the nilreduction of this eigencurve is isomorphic to the Hecke eigencurve which can be defined using Buzzard's eignevariety machine in this case.

In chapter 5, we study the p–adic Hodge theory of our overconvergent modular forms, using the theory of Kisin, Liu and others. We define the finite-slope subspace (à la Kisin) of our eigencurve using the family of pseudo-representations defined over it. We then apply Liu's results to show that the finite–slope subspace of our eigencurve is the whole eigencurve.

Finally, in chapter 6, we draw all the results together and prove that the eigencurve (associated to our Shimura curve) is proper over the weight space.

2. Automorphic Forms over unitary Shimura curves

2.1. **Shimura curves.** In the first part of this chapter, we will set up the field of definition F and the quaternion algebra B used to define our Shimura curves. We will follow the work of Carayol in [Car86a] and indicate any differences when they arise.

2.1.1. *The setup.* Fix a prime $p \neq 2$ and a totally real number field F. Let $d = [F : \mathbb{Q}] > 1$. Write $\eta_1, \ldots, \eta_d$ for the real embeddings of F. Also, fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$. Write $\mathcal{P}_1, \ldots, \mathcal{P}_m$ for the prime ideals of $\mathcal{O}_F$ above p, and define $F_{\mathcal{P}_i}$ to be the completion of F at $\mathcal{P}_i$, for any i.

Set $\mathcal{P} := \mathcal{P}_1$ and $\eta := \eta_1$. Also, set $\mathcal{O}_{\mathcal{P}} := \mathcal{O}_{F_{\mathcal{P}}}$ and fix ϖ to be a uniformiser of $\mathcal{O}_{\mathcal{P}}$. Denote the residue field $\mathcal{O}_{\mathcal{P}}/\varpi$ by κ, and in everything that follows, we will assume $q := |\kappa| > 3$.

There is a valuation $v_{\mathcal{P}}(\cdot)$ on $F_{\mathcal{P}}$ (extending the one on $\mathbb{Q}_p$), which we normalise to ensure $v_{\mathcal{P}}(\varpi) = 1$. This gives a norm $|.|$ on $F_{\mathcal{P}}$, where $|x| := q^{-v_{\mathcal{P}}(x)}$.

Now let B be a quaternion algebra over F which satisfies the following properties:

- B splits at η and is ramified at $\eta_2, \ldots, \eta_d$
- $B \otimes_F F_{\mathcal{P}} \cong M_2(F_{\mathcal{P}})$.

Choose any negative rational λ such that p splits in $\mathbb{Q}(\sqrt{\lambda})$, and define $E := F(\sqrt{\lambda})$. We can regard E as a subfield of $\mathbb{C}$ using the real embedding η as follows:

$$x + y\sqrt{\lambda} \longmapsto \eta(x) + \eta(y)\sqrt{\lambda} \qquad \forall x, y \in F.$$

If we fix $\mu \in \mathbb{Q}_p$ to be a squareroot of λ, then (see [Bra12] §1.3.2 equation (1.3.2)) we can view $F_{\mathcal{P}}$ as an E–algebra. This uses the isomorphisms:

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} (F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times (F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$$

and

$$(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \xrightarrow{\sim} (F_{\mathcal{P}_1} \oplus \ldots \oplus F_{\mathcal{P}_m}),$$

whence the E–algebra structure is then defined via the composition:

$$E \to E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} (F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times (F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \to (F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \xrightarrow{\sim} (F_{\mathcal{P}_1} \oplus \ldots \oplus F_{\mathcal{P}_m}) \to F_{\mathcal{P}_1}$$

which maps

$$\sqrt{\lambda} \mapsto \sqrt{\lambda} \otimes 1 \mapsto (\mu \otimes 1, -\mu \otimes 1) \mapsto \mu \otimes 1 \mapsto \mu \mapsto \mu$$

We will define sets $S_{\mathbb{Q}}^{\lambda}$ and $S_E^{D,\lambda}$ for use later in §2.4. Define $S_{\mathbb{Q}}^{\lambda}$ to be the set of all rational primes that split in $\mathbb{Q}(\sqrt{\lambda})$. For each $l \in S_{\mathbb{Q}}^{\lambda}$, choose an element $\mu_l \in \mathbb{Q}_l$ which is a square root of λ, and in particular choose μ_p to be μ.

Let $l \in S_{\mathbb{Q}}^{\lambda}$ and $\mathfrak{l}$ be a prime of $\mathcal{O}_F$ above l. If $\mathfrak{L}$ is a prime of $\mathcal{O}_E$ above $\mathfrak{l}$, then $E_{\mathfrak{L}} \cong F_{\mathfrak{l}}$, where $E_{\mathfrak{L}}$ is the completion of E at $\mathfrak{L}$ and $F_{\mathfrak{l}}$ is the completion of F at $\mathfrak{l}$.

If we have a map $E \to F_{\mathfrak{l}}$, then this gives a map $\mathcal{O}_E \to \mathcal{O}_{F_{\mathfrak{l}}}$, and the pullback of the maximal ideal of $\mathcal{O}_{F_{\mathfrak{l}}}$ gives a prime of $\mathcal{O}_E$ lying above $\mathfrak{l}$.

Since E is the compositum of F and $\mathbb{Q}(\sqrt{\lambda})$, and any map $E \to F_{\mathfrak{l}}$ fixes F, there are only two possible maps $E \to F_{\mathfrak{l}}$, each determined by the image of $\sqrt{\lambda}$ (because $\sqrt{\lambda}$ must be sent to μ_l or $-\mu_l$).

For the map $E \to F_{\mathfrak{l}}$ obtained from sending $\sqrt{\lambda}$ to μ_l, call the resulting prime of $\mathcal{O}_E$, $\mathfrak{L}$. For the map where $\sqrt{\lambda}$ is sent to $-\mu_l$, call the corresponding prime of $\mathcal{O}_E$, $\overline{\mathfrak{L}}$.

Then, we define the set $S_E^{D,\lambda}$ to be the set of all primes $\mathfrak{Q}$ of $\mathcal{O}_E$ with the following properties:

(1) $\mathfrak{Q}$ lies over the rational prime q, such that $q \in S_{\mathbb{Q}}^{\lambda}$

(2) for any prime $\mathfrak{Q}$ of $\mathcal{O}_E$ lying above q, we have that

$$\mathcal{O}_D \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathfrak{Q}}} \cong M_2(\mathcal{O}_{E_{\mathfrak{Q}}}),$$

where $E_{\mathfrak{Q}}$ denotes the completion of E at $\mathfrak{Q}$, and $\mathcal{O}_D := \mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_E$.

Let $D := B \otimes_F E$ be a quaternion algebra over E. If $\{1, i, j, k\}$ is a Quaternion basis of B ($i^2, j^2 \in F^\times$, $ij = k = -ji$), then the canonical involution of B is defined as

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto x_0 = x_1 i - x_2 j - x_3 k$$

for any $x_0, x_1, x_2, x_3 \in F$ and is denoted by $'$. If $z \mapsto \overline{z}$ is the nontrivial element of $\mathrm{Gal}(E/F)$, then we can define an involution on D as follows:

$$l = b \otimes_F z \;\longmapsto\; b' \otimes_F \overline{z} = \overline{l}.$$

Furthermore, by choosing elements $\alpha, \delta \in D$ such that $\alpha = -\overline{\alpha}$ and $\delta = \overline{\delta}$, one can define a second involution * on D by

$$l^* := \delta^{-1} \overline{l} \delta$$

and a symplectic bilinear form θ on D as:

$$\theta(v, w) := \mathrm{Tr}_{E/\mathbb{Q}}(\alpha \cdot \mathrm{Tr}_{D/E}(v \delta w^*)).$$

Choose a maximal order $\mathcal{O}_B$ of B and an isomorphism $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathcal{P}} \cong M_2(\mathcal{O}_{\mathcal{P}})$. Write $\mathcal{O}_D$ for the maximal order of D which corresponds to $\mathcal{O}_B$ (i.e. $\mathcal{O}_D := \mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_E$). The earlier isomorphism

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} (F_{\mathcal{P}_1} \oplus \ldots \oplus F_{\mathcal{P}_m}) \oplus (F_{\mathcal{P}_1} \oplus \ldots \oplus F_{\mathcal{P}_m})$$

gives rise to a decomposition

$$\begin{array}{ccccccccccc}
\mathcal{O}_D \otimes \mathbb{Z}_p & \xrightarrow{\sim} & \mathcal{O}_{D_1^1} & \oplus \cdots \oplus & \mathcal{O}_{D_m^1} & \oplus & \mathcal{O}_{D_1^2} & \oplus \cdots \oplus & \mathcal{O}_{D_m^2} \\
\bigcap & & \bigcap & & \bigcap & & \bigcap & & \bigcap \\
D \otimes \mathbb{Q}_p & \xrightarrow{\sim} & D_1^1 & \oplus \cdots \oplus & D_m^1 & \oplus & D_1^2 & \oplus \cdots \oplus & D_m^2
\end{array}$$

where each D_j^k is an $F_{\mathcal{P}_j}$-algebra isomorphic to $B \otimes_F F_{\mathcal{P}_j}$. Additionally, $\mathcal{O}_D$ can be chosen in such a way that each $\mathcal{O}_{D_j^k}$ is a maximal order of D_j^k, and such that $\mathcal{O}_{D_1^2} \subset D_1^2 = M_2(F_{\mathcal{P}})$ is identified with $M_2(\mathcal{O}_{\mathcal{P}})$.

Therefore, any $\mathcal{O}_D \otimes \mathbb{Z}_p$-module M possesses an analagous decomposition:

$$M \xrightarrow{\sim} M_1^1 \oplus \cdots \oplus M_m^1 \oplus M_1^2 \oplus \cdots \oplus M_m^2$$

where each M_j^k is an $\mathcal{O}_{D_j^k}$-module, and $\mathcal{O}_D$ acts through its image in $\mathcal{O}_{D_j^k}$. Note that M_1^2, which is an $M_2(\mathcal{O}_{\mathcal{P}})$-module, breaks up even further into $M_1^{2,1} \oplus M_1^{2,2}$ upon choosing orthogonal idempotents e, f in $M_2(\mathcal{O}_{\mathcal{P}})$.

2.1.2. *The Shimura curves over* $\mathbb{C}$. Define G to be the algebraic group over $\mathbb{Q}$ such that, for any $\mathbb{Q}$–algebra R,

$$G(R) = \{D - \text{linear symplectic similitudes of } (D \otimes_{\mathbb{Q}} R, \theta \otimes_{\mathbb{Q}} R)\}.$$

Here, a symplectic similitude is any bilinear map $\psi : (D \otimes_{\mathbb{Q}} R) \times (D \otimes_{\mathbb{Q}} R) \to D \otimes_{\mathbb{Q}} R$ such that for any $\sigma \in \mathrm{GL}(D)$ (D considered as a $\mathbb{Q}$-vector space) there exists a constant $m_\sigma \in \mathbb{Q}$ satisfying $\psi(\sigma(x), \sigma(y)) = m_\sigma \psi(x, y)$.

Let $\mathbb{A}^f$ denote the ring of finite adèles over $\mathbb{Q}$, and let $\mathbb{A}^{f,p}$ denote the finite adèles over $\mathbb{Q}$ away from p. By Exemple 2.6.3 of [Car86a] there is a decomposition

$$G(\mathbb{A}^f) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_2(F_{\mathcal{P}}) \times (B \otimes_F F_{\mathcal{P}_2})^\times \times \cdots (B \otimes_F F_{\mathcal{P}_m})^\times \times G(\mathbb{A}^{f,p}).$$

Denote by Γ the product $(B \otimes_F F_{\mathcal{P}_2})^\times \times \cdots (B \otimes_F F_{\mathcal{P}_m})^\times \times G(\mathbb{A}^{f,p})$.

The subgroups of G that we will be working with will have the assumed form $K = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H$, where H is a compact open subgroup of Γ, and $K_{\mathcal{P}}$ is a compact open subgroup of $\mathrm{GL}_2(F_{\mathcal{P}})$. We will later restrict to certain cases of $K_{\mathcal{P}}$ and H.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. The $\mathbb{R}$-points of $\mathrm{Res}_{F/\mathbb{Q}}(B^*)$ form a group isomorphic to $\mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H})^{d-1}$ ($\mathbb{H}$ denoting the real quaternions), and there is a map $h_1 : \mathbb{S} \to (\mathrm{Res}_{F/\mathbb{Q}}(B^*))_{\mathbb{R}}$ which, on $\mathbb{R}$-points, sends

$$x + yi \mapsto \left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, 1, \ldots, 1 \right].$$

Similarly, the $\mathbb{R}$-points of $(\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m))_{\mathbb{R}}$ are isomorphic to $(\mathbb{C}^\times)^d$, and there is a map $h_2 : \mathbb{S} \to (\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m))_{\mathbb{R}}$ which on $\mathbb{R}$-points sends

$$z \mapsto \left(1, z^{-1}, \ldots, z^{-1}\right).$$

Combining these, we have a projection

$$h_1 \times h_2 : \mathbb{S} \longrightarrow (\mathrm{Res}_{F/\mathbb{Q}}(B^*) \times \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m))_{\mathbb{R}}$$

which factors through a morphism $h' : \mathbb{S} \mapsto G_{\mathbb{R}}$.

If we define

$$X := \text{the } G(\mathbb{R}) - \text{conjugacy class of } h'$$

then X can be identified with the usual Poincaré upper–half plane, and we can form the double quotient

$$M_K(\mathbb{C}) := G(\mathbb{Q})\backslash(G(\mathbb{A}^f) \times X)/K.$$

Here, K acts on $G(\mathbb{A}^f)$ by right multiplication, and $G(\mathbb{Q})$ acts on X by conjugation. This double quotient is actually a compact Riemann surface, and the next step is to give it the structure of an algebraic variety over E.

In [Shi70] it is shown how to define a canonical model of this Riemann surface over E which is smooth and proper. By base changing to $F_{\mathcal{P}}$ (viewing $F_{\mathcal{P}}$ as an E–algebra as mentioned earlier), one can show that the resulting scheme over $F_{\mathcal{P}}$ also represents the moduli problem given in definition 2.1.

2.1.3. *The moduli problem over $F_{\mathcal{P}}$.*

Definition 2.1. Define a functor F from the category of Noetherian $F_{\mathcal{P}}$–schemes to the category of sets as follows: for an $F_{\mathcal{P}}$–scheme S, define $F(S)$ to be the set of isomorphism classes of 4–tuples $(A, i, \theta, \overline{\alpha})$, where

(1) A is an abelian scheme of relative dimension $4d$ over S, with $i : \mathcal{O}_D \to \mathrm{End}_S(A)$ satisisyfing
   - the projective $\mathcal{O}_S$–module $\mathrm{Lie}(A)_1^{2,1}$ has rank 1, and $\mathcal{O}_{\mathcal{P}}$ acts on it via the inclusion of $\mathcal{O}_{\mathcal{P}}$ in $\mathcal{O}_S$
   - for $j \geqslant 2$, we have $\mathrm{Lie}(A)_j^2 = 0$.

(2) θ is a polarisation, whose degree is prime to p, and such that the corresponding Rosati involution sends $i(l)$ to $i(l^*)$

(3) $\overline{\alpha}$ is the K–level structure, i.e. a class mod K of a symplectic $\mathcal{O}_D$–linear isomorphism

$$\widehat{T}(A) \xrightarrow{\sim} \mathcal{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$$

From §2.6 of [Car86a], the functor F is representable by an $F_{\mathcal{P}}$–scheme which is denoted $M(K)$, and is referred to as the Shimura curve of level K.

For the level structure in part (3), $\widehat{T}(A)$ means the product (over all primes l) of the Tate modules $T_l(A)$, considered as an étale sheaf. The symplectic form comes from the Weil pairing composed with θ.

The level structure $\overline{\alpha}$ can be made more explicit by choosing $K_{\mathcal{P}}$ specifically. Let $K = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H$ as before, and let $n \geqslant 0$ be an integer. Consider the following two choices for $K_{\mathcal{P}}$.

(1) $K_0(\varpi^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) : c \equiv 0 \bmod \varpi^n \right\}$

(2) $K_1(\varpi^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) : a \equiv 1 \bmod \varpi^n, c \equiv 0 \bmod \varpi^n \right\}$

Define $\widehat{T}^p(A) := \prod_{l \neq p} T_l(A)$ and $\widehat{\mathbb{Z}}^p := \prod_{l \neq p} \mathbb{Z}_l$. Also, write

$$\begin{array}{rcl} T_p^{\mathcal{P}}(A) & := & T_p(A)_2^2 \oplus \cdots \oplus T_p(A)_m^2 \\ W_p^{\mathcal{P}} & := & \mathcal{O}_{D_2^2} \oplus \cdots \oplus \mathcal{O}_{D_m^2} \\ \widehat{W}^p & := & \mathcal{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \end{array}$$

Here are some explicit descriptions for the level structures. Define $A[\varpi^n]_1^{2,1}$ to be the ϖ^n–torsion in $A[p^n]_1^{2,1}$

(1) Let $K_{\mathcal{P}} = K_0(\varpi^n)$. A level structure $\overline{\alpha}$ for $K = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H$ is equivalent to giving a pair $(\overline{\alpha}^{\mathcal{P}}, C)$ where:
   - C is a finite–flat subgroup of rank q^n of $A[\varpi^n]_1^{2,1}$, which is invariant under the action of $\mathcal{O}_{\mathcal{P}}$
   - $\overline{\alpha}^{\mathcal{P}}$ is a class of isomorphisms $\alpha_p^{\mathcal{P}} \oplus \alpha^p : T_p^{\mathcal{P}}(A) \oplus \widehat{T}^p(A) \xrightarrow{\sim} W_p^{\mathcal{P}} \oplus \widehat{W}^p$ modulo H, where $\overline{\alpha}^{\mathcal{P}}$ is linear and α^p is symplectic.

(2) Let $K_{\mathcal{P}} = K_1(\varpi^n)$. A level structure for $K = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H$ is equivalent to giving a pair $(\overline{\alpha}^{\mathcal{P}}, Q)$, where $\overline{\alpha}^{\mathcal{P}}$ is as in (1) and Q is a point of exact order ϖ^n in $A[\varpi^n]_1^{2,1}$.

When $K_{\mathcal{P}} = K_0(\varpi^n)$ and $K_{\mathcal{P}} = K_1(\varpi^n)$, we denote the corresponding Shimura curves by $M(H, \varpi^n)$ and $M(H\varpi^n)$ respectively. When $n = 0$, $K_0(\varpi^n) = K_1(\varpi^n)$ and the level structure $\overline{\alpha}$ depends only on $\alpha_p^{\mathcal{P}}$ and α^p. We then have that $K = \mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) \times H$, and the corresponding Shimura curve will be denoted $M(H)$.

We shall say that $M(H)$ is the Shimura curve of level H, $M(H, \varpi^n)$ is the Shimura curve of level $H \times K_0(\varpi^n)$ and $M(H\varpi^n)$ is the Shimura curve of level $H \times K_1(\varpi^n)$. Each of these is an $F_{\mathcal{P}}$–scheme. Since each of these schemes is a representing object of the moduli problem 2.1, they all come with a universal abelian scheme, denoted A, and a map ϵ to the corresponding Shimura curve.

There are also forgetful maps between Shimura curves of different levels, defined via transformations of functors. If $m \geqslant n$ are natural numbers, then there are forgetful morphisms:

$$M(H\varpi^m) \longrightarrow M(H\varpi^n)$$

$$M(H,\varpi^m) \longrightarrow M(H,\varpi^n)$$

$$M(H\varpi^n) \longrightarrow M(H,\varpi^n)$$

$$M(H\varpi^n) \longrightarrow M(H)$$

$$M(H,\varpi^n) \longrightarrow M(H)$$

The last two in particular will be used later in defining overconvergent modular forms.

2.1.4. *The moduli problem over $\mathcal{O}_{\mathcal{P}}$.*

Definition 2.2. For H as above which is 'small enough', the Shimura curve of level H has a *smooth, proper integral* model over $\mathcal{O}_{\mathcal{P}}$, denoted $\mathcal{M}(H)$. By §5.1 of [Car86a], $\mathcal{M}(H)$ represents the following moduli problem. (See *loc. cit.* for the definition of small enough).

Define a functor F from the category of Noetherian $\mathcal{O}_{\mathcal{P}}$–schemes to the category of sets as follows: for an $\mathcal{O}_{\mathcal{P}}$–scheme S, define $F(S)$ to be the set of isomorphism classes of 4–tuples $(A, i, \theta, \overline{\alpha})$, where

(1) A is an abelian scheme of relative dimension $4d$ over S, with $i : \mathcal{O}_D \to \mathrm{End}_S(A)$ satisyfing
   - the projective $\mathcal{O}_S$–module $\mathrm{Lie}(A)_1^{2,1}$ has rank 1, and $\mathcal{O}_{\mathcal{P}}$ acts on it via the inclusion of $\mathcal{O}_{\mathcal{P}}$ in $\mathcal{O}_S$
   - for $j \geqslant 2$, we have $\mathrm{Lie}(A)_j^2 = 0$.

(2) θ is a polarisation, whose degree is prime to p, and such that the corresponding Rosati involution sends $i(l)$ to $i(l^*)$

(3) $\overline{\alpha}$ is the K–level structure, i.e. a pair $(\alpha_p^{\mathcal{P}}, \alpha^p)$ modulo H, where:
   - $\alpha_p^{\mathcal{P}} : T_p^{\mathcal{P}}(A) \xrightarrow{\sim} W_p^{\mathcal{P}}$ is a linear isomorphism

- $\alpha^p : \widehat{T}^p(A) \xrightarrow{\sim} \widehat{W}^p$ is a symplectic isomorphism.

2.1.5. *The sheaf* $\underline{\omega}$. Let $\epsilon : \mathcal{A} \to \mathcal{M}(H)$ be the natural map from the universal abelian variety to the Shimura curve of level H. Consider $\Omega^1_{\mathcal{A}/\mathcal{M}(H)}$, the sheaf of relative differentials on $\mathcal{A}$. The push–forward

$$(\dagger) \quad \epsilon_* \Omega^1_{\mathcal{A}/\mathcal{M}(H)}$$

is an $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$–module. Thus, we can decompose $(\dagger)$ as an $\mathcal{O}_D \otimes \mathbb{Z}_p$–module, and take its $-^{2,1}_{1}$ part.

Definition 2.3. (1) Define the sheaf

$$\underline{\omega}_{H,\mathcal{O}_{\mathcal{P}}} := (\epsilon_* \Omega^1_{\mathcal{A}/\mathcal{M}(H)})^{2,1}_{1}.$$

Let $\underline{\omega}_H$ be the base change of this sheaf via $\mathcal{O}_{\mathcal{P}} \to F_{\mathcal{P}}$ defined on $M(H)$.

(2) Let $\pi : M(H, \varpi^n) \longrightarrow M(H)$ be the forgetful map. Define

$$\underline{\omega}_{H \times K_0(\varpi^n)} := \pi^* \underline{\omega}_H.$$

(3) Let $\pi : M(H\varpi^n) \longrightarrow M(H)$ be the forgetful map. Define

$$\underline{\omega}_{H \times K_1(\varpi^n)} := \pi^* \underline{\omega}_H.$$

Often we will just write $\underline{\omega}$ for the sheaf, without reference to the groups $K_{\mathcal{P}}$ and H. Note that part (1) of the moduli problems 2.1 and 2.2 ensure that $\underline{\omega}$ is an invertible sheaf.

2.2. **Classical automorphic forms.** We are now ready to begin defining the spaces of modular forms that we will be using. First we will define those of integral weight, which exist as global sections of the sheaves $\underline{\omega}^{\otimes k}$ on our Shimura curves.

Recall our choice of prime-to-$\mathcal{P}$ level structure H from §2.1.2. If R is an $\mathcal{O}_{\mathcal{P}}$-algebra, we let $\mathcal{M}(H)_R$ denote the base change of the Shimura curve $\mathcal{M}(H)$ from $\mathcal{O}_{\mathcal{P}}$ to R. If R is an $F_{\mathcal{P}}$-algebra, we use the same convention for $M(H, \varpi^n)_R$ and $M(H\varpi^n)_R$.

We will denote the corresponding pullbacks of $\underline{\omega}$ by $\underline{\omega}$ again.

Definition 2.4. Let R be an $\mathcal{O}_{\mathcal{P}}$–algebra and $k \in \mathbb{Z}$. We define

$$S^D(R, H, k) := H^0(\mathcal{M}(H)_R, \underline{\omega}^{\otimes k})$$

to be the space of modular forms of integral weight k, level H, coefficients in R and with respect to the quaternion algebra D. Now let R be an $F_{\mathcal{P}}$–algebra. There are analagous definitions for weight k modular forms of level $H \times K_0(\varpi^n)$ and $H \times K_1(\varpi^n)$:

$$S^D(R, H \times K_0(\varpi^n), k) := H^0(M(H, \varpi^n)_R, \underline{\omega}^{\otimes k})$$

$$S^D(R, H \times K_1(\varpi^n), k) := H^0(M(H\varpi^n)_R, \underline{\omega}^{\otimes k}).$$

The goal of the rest of this chapter is to define overconvergent modular forms of general (i.e. integral *and* non–integral) weight. Before we can do this, we must define certain admissible open subspaces of the Shimura curves, and this is done using a specific modular form called the Hasse invariant. This will live inside $S^D(\kappa, H, k)$, where we recall that $\kappa = \mathcal{O}_{\mathcal{P}}/\varpi$.

2.2.1. *The Hasse invariant.* Consider the curve $\mathcal{M}(H)_\kappa$, the Shimura curve basechanged to κ, and let $\mathrm{Spec}(R) \subset \mathcal{M}(H)_\kappa$ be an open affine subset. By the moduli problem 2.2, this gives an abelian variety $\mathcal{A}$ which is defined over $\mathrm{Spec}(R)$. Since R is a κ–algebra, the q–th power map $x \mapsto x^q$ on R gives us a morphism $\sigma_q : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$, and we can define the scheme $\mathcal{A}^{(q)}$ as the fibre product

$$\begin{array}{ccc} \mathcal{A}^{(q)} & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \stackrel{\sigma_q}{\to} & \mathrm{Spec}(R) \end{array}$$

We can also consider the map $\sigma_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ which is the identity map on the underlying topological space, and the q–th power map $a \mapsto a^q$ on $\mathcal{O}_{\mathcal{A}}$. Therefore, there is a natural map $F_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{(q)}$, the relative Frobenius map.

On the other hand, we can apply the above construction to the dual scheme $\mathcal{A}^*$ and get a map $F_{\mathcal{A}^*}$. However, $(\mathcal{A}^*)^{(q)}$ is isomorphic to $(\mathcal{A}^{(q)})^*$, and hence there is a map $\mathcal{A}^* \to (\mathcal{A}^{(q)})^*$. The dual of this map is denoted Ver, and is called the Verschiebung map. Thus $\mathrm{Ver} : \mathcal{A}^{(q)} \to \mathcal{A}$.

Now recall that we have the sheaf $\underline{\omega}$ on $\mathcal{M}(H)$. Let $\underline{\omega}_{\mathcal{A}}$ and $\underline{\omega}_{\mathcal{A}^{(q)}}$ be the pullbacks of this sheaf to $\mathcal{A}$ and $\mathcal{A}^{(q)}$ respectively, so that Verschiebung gives us a morphism

$$\underline{\omega}_{\mathcal{A}} \longrightarrow \underline{\omega}_{\mathcal{A}^{(q)}}$$

The right–hand side is just $\underline{\omega}_{\mathcal{A}}^{\otimes q}$ because $\kappa = \mathbb{F}_q$, the finite field with q elements, thus we have a distinguished element $f \in \mathrm{Hom}(\underline{\omega}_{\mathcal{A}}, \underline{\omega}_{\mathcal{A}}^{\otimes q})$. But $\underline{\omega}_{\mathcal{A}}$ is an invertible sheaf, so by tensoring with its dual, we can also say $f \in \mathrm{Hom}(\mathcal{O}_{\mathcal{A}}, \underline{\omega}_{\mathcal{A}}^{\otimes(q-1)})$. In particular, f is completely determined by the image of the global section 1, and we define $\mathbf{H}|_{\mathrm{Spec}(R)} := f(1)$ to be that image. As we run through all the open affines $\mathrm{Spec}(R)$ of $\mathcal{M}(H)_\kappa$, we obtain a collection of local sections $\mathbf{H}|_{\mathrm{Spec}(R)}$. These glue together (because Verschiebung is compatible with base change and fibre products), producing a global section $\mathbf{H} \in H^0(\mathcal{M}(H)_\kappa, \underline{\omega}^{\otimes(q-1)})$.

Definition 2.5. $\mathbf{H}$ is referred to as the Hasse invariant. It is a modular form of weight $q-1$ and level H, with coefficients in κ.

By Proposition 6.2 of [Kas04], since $q > 3$ the Hasse invariant can be lifted to an element of $H^0(\mathcal{M}(H), \underline{\omega}^{\otimes(q-1)})$, i.e. to characteristic 0. Choose once and for all a lift of $\mathbf{H}$, and denote it by E_{q-1}.

2.2.2. *Rigid analytic constructions.* Let $\mathfrak{M}(H)$ (resp. $\mathfrak{M}(H, \varpi^n), \mathfrak{M}(H\varpi^n)$) be the rigidification of the Shimura curve $M(H)$ (resp. $M(H, \varpi^n), M(H\varpi^n)$). We are going to define admissible open subspaces of these rigid curves, which will let us define overconvergent modular forms.

Consider the Hasse invariant $\mathbf{H} \in S^D(\kappa, H, q-1)$. Cover the Shimura curve $\mathcal{M}(H)$ by open affines U_i such that the sheaf $\underline{\omega}$ becomes trivial on each U_i. Therefore, $\mathcal{M}(H)_\kappa$ is covered by the open affines $U_{i\kappa}$. For each i, we can write $\mathbf{H}|_{U_{i\kappa}} = a_i\omega_i^{\otimes(q-1)}$ for some element $a_i \in \mathcal{O}_{U_{i\kappa}}(U_{i\kappa})$ and section ω_i of $\underline{\omega}(U_i)$. Lift each a_i to an element $\tilde{a}_i \in \mathcal{O}_{U_i}(U_i)$. A valuation can be defined on the closed points of $\mathfrak{M}(H)$, which does not depend on the choice of U_i, ω_i or the lifts $\tilde{a}_i$.

Definition 2.6 (Valuation on $\mathfrak{M}(H)$)**.** Let $x \in \mathfrak{M}(H)$ be a point. It is given by a morphism $\mathrm{Spec}(L_x) \to \mathfrak{M}(H)$ for some field L_x, and since $\mathcal{M}(H)$ is proper, the valuative criterion of properness implies that x arises from a morphism $\mathrm{Spec}(\mathcal{O}_{L_x}) \to \mathcal{M}(H)$. For some i, this morphism factors through U_i. Let $\tilde{a}_i(x)$ denote the image of $\tilde{a}_i$ in $\mathcal{O}_{L_x}$, and define

$$v(x) := \min(v(\tilde{a}_i(x)), 1)$$

where the valuation on the right–hand side is the one on $\mathcal{O}_{L_x}$ (and extends the valuation on $\mathcal{O}_{\mathcal{P}}$).

Definition 2.7 (The space $\mathfrak{M}(H)(w)$)**.** Let $0 \leqslant w < \frac{q}{q+1}$ be a rational number. Define $\mathfrak{M}(H)(w)$ to be the admissible open space, whose points are all $x \in \mathfrak{M}(H)$ satisfying $v(x) \leqslant w$.

Now we will construct the spaces $\mathfrak{M}(H, \varpi^n)(w)$, for all $n \in \mathbb{N}$, where $0 \leqslant w < \frac{1}{q^{n-2}(q+1)}$. Recall that there is a forgetful map $\pi : \mathfrak{M}(H, \varpi^n) \to \mathfrak{M}(H)$. We don't want to consider all of $\pi^{-1}(\mathfrak{M}(H)(w))$ though, because it has several connected components. Instead, we shall only consider the connected component where the points have level structure involving the n–th canonical subgroup.

2.2.3. *The canonical subgroup of an abelian variety.* Let $n \in \mathbb{N}$ and w be a rational number such that $0 \leqslant w < \frac{1}{q^{n-2}(q+1)}$. Let $(A, i, \theta, \overline{\alpha})$ be a point of $\mathfrak{M}(H)(w)$ (as given in the moduli problem 2.1). Then there is a finite flat subgroup scheme C of A possessing the following properties (amongst other things):

(1) C has rank q^{4d}, is stable under the action of $\mathcal{O}_D$, and lives in $A[\varpi]$

(2) C is of *type 1*, which is to say that in the $\mathcal{O}_D \otimes \mathbb{Z}_p$ decomposition of C, we have that $C_2^2 \oplus \ldots \oplus C_m^2 = 0$.

This subgroup C is called the first canonical subgroup of A. If $n \geqslant 2$, then we can define the n–th canonical subgroup C_n of A inductively, as the kernel of the composition

$$A \to A/C \to (A/C)/C'_{n-1}$$

where C'_{n-1} is the $(n-1)$–th canonical subgroup of A/C. We have that $C_n \subset A[\varpi^n]$.

Where there is possible confusion in the subscripts, the outermost subscript and superscript will refer to the $\mathcal{O}_D \otimes \mathbb{Z}_p$ decomposition, so for example $(C_2)^{2,1}_1$ means the $-^{2,1}_1$ part of the second canonical subgroup.

Definition 2.8. Define $\mathfrak{M}(H, \varpi^n)(w)$ to be the set of all points $(A, i, \theta, \overline{\alpha})$ in $\pi^{-1}(\mathfrak{M}(H)(w))$, such that the finite–flat subgroup (given by $\overline{\alpha}$) is the n–th canonical subgroup of A. Furthermore, if $\pi : \mathfrak{M}(H\varpi^n) \to \mathfrak{M}(H, \varpi^n)$ is the forgetful map, we define $\mathfrak{M}(H\varpi^n)(w) := \pi^{-1}(\mathfrak{M}(H, \varpi^n)(w))$.

The rigidification of $M(H)$ to $\mathfrak{M}(H)$ also sends the invertible sheaf $\underline{\omega}$ to a sheaf on $\mathfrak{M}(H)$; hence we get an invertible sheaf on $\mathfrak{M}(H)$ and the corresponding pullbacks, which will be denoted $\underline{\omega}$ again. We can now define weight k overconvergent modular forms as the sections of $\underline{\omega}^{\otimes k}$ over these subspaces.

Definition 2.9. Let L be a complete, nonarchimedean field extension of $F_{\mathcal{P}}$ and let $0 \leqslant w < \frac{q}{q+1}$ be rational. Define

$$S^D(L, H, k, w) := H^0(\mathfrak{M}(H)(w)\hat{\otimes}_{F_{\mathcal{P}}} L, \underline{\omega}^{\otimes k})$$

to be the space of w–overconvergent modular forms of integral weight k, level H, having coefficients in L and with respect to the quaternion algebra D.
If $n \in \mathbb{N}$ and $0 \leqslant w < \frac{1}{q^{n-2}(q+1)}$, then there are analagous definitions

$$S^D(L, H \times K_0(\varpi^n), k, w) := H^0(\mathfrak{M}(H, \varpi^n)(w)\hat{\otimes}_{F_{\mathcal{P}}} L, \underline{\omega}^{\otimes k})$$

$$S^D(L, H \times K_1(\varpi^n), k, w) := H^0(\mathfrak{M}(H\varpi^n)(w)\hat{\otimes}_{F_{\mathcal{P}}} L, \underline{\omega}^{\otimes k})$$

for w–overconvergent modular forms of weight k, level $H \times K_0(\varpi^n)$ (resp. $H \times K_1(\varpi^n)$), having coefficients in L and w.r.t the quaternion algebra D.

Definition 2.10. Define

$$S^{\dagger,D}(L,H,k) := \varinjlim_{w>0} S^D(L,H,k,w)$$

$$S^{\dagger,D}(L,H\times K_0(\varpi^n),k) := \varinjlim_{w>0} S^D(L,H\times K_0(\varpi^n),k,w)$$

$$S^{\dagger,D}(L,H\times K_1(\varpi^n),k) := \varinjlim_{w>0} S^D(L,H\times K_1(\varpi^n),k,w)$$

as the space of all overconvergent modular forms of weight k and level H (resp. $H \times K_0(\varpi^n)$ and $H \times K_1(\varpi^n)$) with coefficients in L.

2.2.4. *The weight space.* In order to define modular forms of general weight, we have to define the weights that we are interested in.

Recall that the kernel of the canonical map $\mathcal{O}_\mathcal{P} \to \mathcal{O}_\mathcal{P}/\varpi$ is $(1+\varpi\mathcal{O}_\mathcal{P})$. A section of this map is given by the Teichmuller character, and hence we have an isomorphism (after taking unit groups)

$$\mathcal{O}_\mathcal{P}^\times \cong (\mathcal{O}_\mathcal{P}/\varpi)^\times \times (1+\varpi\mathcal{O}_\mathcal{P}).$$

Let $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $s \in K$. For $x \in \mathcal{O}_\mathcal{P}^\times$, write $[x]^i$ for the reduction of x modulo ϖ followed by the application of the Teichmuller character, taken to the i–th power. Also, write $\langle x\rangle^s$ to mean $\exp(s\log(x/[x]))$ (the p–adic logarithm). The isomorphism $\mathcal{O}_\mathcal{P}^\times \cong (\mathcal{O}_\mathcal{P}/\varpi)^\times \times (1+\varpi\mathcal{O}_\mathcal{P})$ is given explicitly as

$$x \mapsto ([x],\langle x\rangle)$$

Definition 2.11 (n–accessible characters)**.** Let K be a complete, nonarchimedean field extension of $F_\mathcal{P}$. Set e to be the ramification degree of the extension $F_\mathcal{P} \supset \mathbb{Q}_p$. Call a character $\chi : \mathcal{O}_\mathcal{P}^\times \to K^\times$ n**–accessible** if there exist three parameters (n,i,s), where

(1) $n \in \mathbb{N}$

(2) $i \in \mathbb{Z}/(q-1)\mathbb{Z}$

(3) $s \in K$ satisfies $v(s) > \frac{e}{p-1} - n$

and χ takes the following form: for all $x \in \mathcal{O}_{\mathcal{P}}^{\times}$ with $v_{\mathcal{P}}(\langle x \rangle - 1) \geqslant n$, we have that

$$\chi(x) = [x]^i \langle x \rangle^s.$$

Note that each $k \in \mathbb{Z}$ gives rise to a 1–accessible character with parameters $i = k \bmod (q-1)$ and $s = k$.

Definition 2.12 (The weight space $\mathcal{W}$)**.** Let K be a complete, nonarchimedean field extension of $F_{\mathcal{P}}$. The weight space is a $F_{\mathcal{P}}$–rigid analytic group whose K–points are defined to be the group of $F_{\mathcal{P}}$–locally analytic characters

$$\mathcal{W}(K) := \mathrm{Hom}(\mathcal{O}_{\mathcal{P}}^{\times}, K^{\times})$$

We see that the decomposition $\mathcal{O}_{\mathcal{P}}^{\times} \cong (\mathcal{O}_{\mathcal{P}}/\varpi)^{\times} \times (1 + \varpi\mathcal{O}_{\mathcal{P}})$ means $\mathcal{W}$ has $q-1$ connected components. If we write $\mathcal{B}$ for the connected component of the identity (the 1–accessible character with parameters $i = s = 0$), then this gives a decomposition $\mathcal{W} = \coprod_{\mathbb{Z}/(q-1)\mathbb{Z}} \mathcal{B}$. In general, $\mathcal{B}$ is a closed subvariety of the d–dimensional open polydisk of radius 1 (see Lemma 3.10 of [ST01]), and $\mathcal{B}$ is 1-dimensional. In §4.1 of [Bra12] Brasca uses this to define an admissible covering $\{\mathcal{W}_n\}_{n \geqslant 1}$ of $\mathcal{W}$, with the property that each point of $\mathcal{W}_n$ is an n–accessible character. In addition to this, each $\mathcal{W}_n$ is an affinoid, which means that if X is any open affinoid of $\mathcal{W}$, we have $X \subset \mathcal{W}_m$ for some $m \in \mathbb{N}$.

2.3. **Overconvergent automorphic forms of general weight.** We now construct sheaves whose global sections will give spaces of overconvergent modular forms of general weight. In order to do this, we require the use of formal models of the rigid Shimura curves we defined previously. Let $\tilde{M}(H)$ denote the formal completion of $\mathcal{M}(H)$ along its special fibre. By rigid-analytic GAGA, the Raynaud rigid analytic generic fibre associated to $\tilde{M}(H)$ is $\mathfrak{M}(H)$ (which we defined to be the rigidification of $M(H)$ in §2.2.2).

Let V be a finite extension of $\mathcal{O}_{\mathcal{P}}$ and w a rational number which is the valuation of an element $\varpi^w \in V$. One can define a formal model $\tilde{M}(H)(w)$ for $\mathfrak{M}(H)(w)_V$ by completing

$$\mathrm{Spec}_{\mathcal{M}(H)_V}(Sym(\omega^{\otimes(q-1)})/(E_{q-1} - \varpi^w)))$$

along its special fibre (where E_{q-1} was fixed after the definition of Hasse invariant). If w and V further satisfy

(1) $0 \leqslant w < \frac{1}{q^{n-2}(q+1)}$

(2) V contains a p–th root of unity, and

(3) V contains a $(q-1)$–th root of $-\varpi$

we can define a formal model $\tilde{M}(H\varpi^n)(w)$ of $\mathfrak{M}(H\varpi^n)(w)_V$ as follows. Let $\{U_i = \mathrm{Spf}(\tilde{A}_i)\}$ be an affine covering of $\tilde{M}(H)(w)$. Since the forgetful morphism $\pi : \mathfrak{M}(H\varpi^n) \to \mathfrak{M}(H)$ is finite-flat, $\pi^{-1}(U_i^{rig})$ is an affinoid $\mathrm{Sp}(B_i)$ in $\mathfrak{M}(H\varpi^n)$. We define $\tilde{B}_i$ to be the normalisation of $\tilde{A}_i$ in B_i. The various $\mathrm{Spf}(\tilde{B}_i)$ glue together to construct $\tilde{M}(H\varpi^n)(w)$. The sheaf $\underline{\omega}$ extends to all of these spaces, and will be denoted by the same notation.

In the following, we prepare for the definition of the Hodge-Tate sequence. Choose an open covering $\{\mathrm{Spf}(R_i)\}_{i\in I}$ of $\tilde{M}(H)(w)$, such that the pullback of $\underline{\omega}$ to each $\mathrm{Spf}(R_i)$ is trivial. We will work locally from now on, so let $\mathcal{U} := \mathrm{Spf}(R)$ denote one of these open affines and fix a generator ω of (the pullback to $\mathrm{Spf}(R)$

of) $\underline{\omega}$. From the original moduli problem for the Shimura curves, the inclusion $\mathrm{Spf}(R) \hookrightarrow \tilde{M}(H)$ gives a formal abelian scheme A over R.

Take the inverse image of $\mathrm{Spf}(R)$ under the map $\tilde{M}(H\varpi^n)(w) \to \tilde{M}(H)(w)$. This gives an open set $\mathrm{Spf}(S)$, where S is a finite R–algebra.

We continue to use subscripts to denote the base change of objects. For example, $\mathcal{U} = \mathrm{Spf}(R)$ is an open set of $\tilde{M}(H)(w)$, the latter of which is a scheme over the ring V; therefore R is a V–algebra and so $\mathcal{U}_{V[1/p]}$ will denote the base change of $\mathcal{U}$ to the field $V[1/p]$.

Fix a geometric point $\eta = \mathrm{Spec}(L) \to \mathcal{U}$ of $\mathcal{U}$, and define $\mathcal{G} := \pi_1(\mathcal{U}_{V[1/p]}, \eta)$. Also, define $\overline{R}$ to be the direct limit of all normal R–algebras $T \subset L$ for which the extension $T_{V[1/p]}/R_{V[1/p]}$ is finite and etale. We have that $\mathcal{G} \cong \mathrm{Gal}(\overline{R}_{V[1/p]}/R_{V[1/p]})$ and $\mathcal{G}$ acts continuously on $\hat{\overline{R}}$, the ϖ–adic completion of $\overline{R}$. We also set $\mathcal{H}$ to be $\mathrm{Gal}(\overline{R}_{V[1/p]}/S_{V[1/p]})$. (For more details, see §2 of [AIS12]).

We now recall the Lubin–Tate ϖ–divisible group (see §3.3 of [ANT67] for further properties), the ϖ–adic Tate module of a group with an action of $\mathcal{O}_{\mathcal{P}}$, and the notion of duality.

Definition 2.13 (Lubin–Tate ϖ–divisible group)**.** Consider the polynomial $\varpi x + x^q \in R[[x]]$. Since this reduces to $\varpi x (\mathrm{mod}\ x^2)$ and to $x^q (\mathrm{mod}\ \varpi)$, it uniquely defines a formal group law on $R[[x]]$ with

$$[\varpi](x) = \varpi x + x^q$$

This is defined to be the **Lubin–Tate ϖ–divisible group $\mathcal{LT}$**. The $\mathcal{O}_{\mathcal{P}}$ action comes via the action on the Lie algebra (from the algebra map $\mathcal{O}_{\mathcal{P}} \to V \to R$).

Definition 2.14 (ϖ–adic Tate–module)**.** Let G be an abelian group with an $\mathcal{O}_{\mathcal{P}}$–action. Its *$\varpi$–adic Tate module* is the inverse limit

$$\mathrm{T}_{\varpi}(G) := \varprojlim_i G[\varpi^i].$$

If G is a ϖ–divisible group (as it will be later), then we define its ϖ–adic Tate module via the $\overline{R}_{V[1/p]}$–points:

$$\mathrm{T}_{\varpi}(G) := \mathrm{T}_{\varpi}(G(\overline{R_{V[1/p]}})).$$

2.3.1. *The map dlog.* We first recall the classical dlog map from algebraic geometry, following §2 of [AIS12]. Let G be a finite–flat group scheme over R which has an action of $\mathcal{O}_{\mathcal{P}}$, and suppose that $G[\varpi^m] = G$ for some $m \in \mathbb{N}$. Consider the functor F from the category of R–schemes to the category of abelian groups, which is such that if X is an R–scheme, then

$$F(X) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{P}}}(G_X, \mathcal{LT}_X).$$

This functor is in fact representable by a scheme denoted $\check{G}$ and we refer to it simply as *the dual of* G where necessary. In particular, if H is a sub $\mathcal{O}_{\mathcal{P}}$–module of $\mathrm{T}_{\varpi}(\check{G})$, then we can use the above to obtain $H^{\perp}$, which is a sub $\mathcal{O}_{\mathcal{P}}$–module of $T_{\varpi}(G)$.

We now have the notation set up to describe the dlog map. Let W be a normal, Noetherian R–algebra which is also ϖ–torsion free. Let G be a group scheme with an action of $\mathcal{O}_{\mathcal{P}}$, and write $\underline{\omega}_{G/R}$ for its module of invariant differentials. If G is killed by ϖ^i, define a map

$$\mathrm{dlog}_G := \mathrm{dlog}_{G,W} : \check{G}(W_{V[1/p]}) \to \underline{\omega}_{G/R} \otimes_R W/\varpi^i W$$

as follows:

(1) if $x \in \breve{G}(W_{V[1/p]})$, then this extends to a point $x \in \breve{G}(W)$, by normality

(2) representability then produces a group scheme homomorphism $f_x : G \to \mathcal{LT}$

(3) define $\mathrm{dlog}_{G,W}(x) := f_x^*(\mathrm{dT})$ to be the pullback of the canonical differential.

The dlog map is functorial with respect to all its arguments ([Bra12], Lemma 3.1.3).

For our purposes, we will take G to be $A[\varpi^i]_1^{2,1}$, where A is the abelian scheme associated to our chosen affine $\mathrm{Spf}(R) \subset \tilde{M}(H)$. Therefore we have the dlog map

$$\mathrm{dlog}_{i,W} : (\overbrace{A[\varpi^i]_1^{2,1}})(W_{V[1/p]}) \to \underline{\omega} \otimes_R W/\varpi^i W.$$

Take the direct limit over all normal, Noetherian R–algebras $W \subset L$ for which W_K/R_K is finite and étale. Then take the projective limit over all i to get the map

$$\mathrm{dlog}_A : \mathrm{T}_\varpi((\overbrace{A[\varpi^\infty]_1^{2,1}}))(\overline{R_{V[1/p]}}) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \to \underline{\omega} \otimes_R \widehat{\overline{R}}.$$

Similarly, if we use $G = \overbrace{A[\varpi^i]_1^{2,1}}$ and take the direct limit and projective limit in a similar way, then we also get the map

$$\mathrm{dlog}_{\breve{A}} : \mathrm{T}_\varpi((A[\varpi^\infty]_1^{2,1}))(\overline{R_{V[1/p]}}) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \to \underline{\omega} \otimes_R \widehat{\overline{R}}.$$

There is the following isomorphism of $\mathcal{G}$–modules:

$$\mathrm{T}_\varpi((A[\varpi^\infty]_1^{2,1})\breve{\ }) \xrightarrow{\sim} \mathrm{T}_\varpi((A[\varpi^\infty]_1^{2,1})^*(1)$$

where the $(-)(1)$ means that the Galois action is twisted by the Lubin–Tate character, and the * means the dual module. Define $a_A := \mathrm{dlog}^*_{\breve{A}}(1)$. It will appear in the Hodge–Tate sequence described in the next section.

2.3.2. *The Hodge–Tate sequence.*

Definition 2.15 (The Hodge–Tate sequence of A)**.** There is a complex:

$$0 \to \underline{\omega}^*_{\breve{A}/R} \otimes_R \widehat{\overline{R}}(1) \xrightarrow{a_A} \mathrm{T}_{\varpi}((A[\varpi^\infty]_1^{2,1}))(\overline{R_{V[1/p]}}) \otimes_{\mathcal{O}_{\mathcal{P}}} \widehat{\overline{R}} \xrightarrow{\mathrm{dlog}_{\breve{A}}} \underline{\omega}_{A/R} \otimes_R \widehat{\overline{R}} \to 0$$

called *the Hodge–Tate sequence of* A. We have that a_A is injective by Remark 2.4 of [AIS12] (but the sequence is not always exact).

Let $n \in \mathbb{N}$, and let C_n denote the n–th canonical subgroup of A. By Proposition 4.3.3 of [Bra12] one can obtain a $\mathcal{G}$–equivariant isomorphism

$$\mathrm{Im}(\mathrm{dlog}_A)/\varpi^{n-v}\mathrm{Im}(\mathrm{dlog}_A) \xrightarrow{\sim} (\widecheck{(C_n)_1^{2,1}}) \otimes_{\mathcal{O}_{\mathcal{P}}} \overline{R}/\varpi^{n-v}\overline{R}$$

Definition 2.16 (The sheaf $\mathcal{F}$)**.** Recall the Galois group $\mathcal{H} = \mathrm{Gal}(\overline{R}_K/S_K)$. Define the sheaf $\mathcal{F}$ to be the unique sheaf such that, for an affine $\mathrm{Spf}(S) \subset \tilde{M}(H\varpi^n)(w)$ as above, we have

$$\mathcal{F}(\mathrm{Spf}(S)) := \mathrm{Im}(\mathrm{dlog}_A)^{\mathcal{H}}$$

$\mathcal{F}$ satisfies

$$(\dagger) \qquad \mathcal{F}/\varpi^{n-v}\mathcal{F} \xrightarrow{\sim} \widecheck{C_1^{2,1}} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\tilde{M}(H\varpi^n)(w)}/\varpi^{n-v}\mathcal{O}_{\tilde{M}(H\varpi^n)(w)}.$$

Here, C means the n–th canonical subgroup of of the universal abelian variety of $\tilde{M}(H\varpi^n)$ which has been pulled back to $\tilde{M}(H\varpi^n)(w)$.

Definition 2.17 (The sheaf $\mathcal{F}'_{n,v}$)**.** Consider the constant sheaf of sets on $\tilde{M}(H\varpi^n)(w)$, given by the subset of points of $\widetilde{C_1^{2,1}}$ of exact order ϖ^n. This is a subsheaf of the right–hand–side of (†), so the isomorphism lets us identify it with a subsheaf of $\mathcal{F}/\varpi^{n-v}\mathcal{F}$. We can then take its inverse image under the canonical map $\mathcal{F} \to \mathcal{F}/\varpi^{n-v}\mathcal{F}$. That final sheaf is defined to $\mathcal{F}'_{n,v}$.

2.3.3. *The sheaf* $\tilde{\Omega}_w^{\chi}$. Let χ be an n-accessible wight character with associated parameters (i, s). We are now going to define the sheaf $\tilde{\Omega}_w^{\chi}$ on $\tilde{M}(H\varpi^n)(w)$ whose global sections will be the w–overconvergent modular forms of weight χ and level $H \times K_1(\varpi^n)$. Consider the sheaf of abelian groups

$$\mathcal{S}_{n,v} := \mathcal{O}_{\mathcal{P}}^{\times}\left(1 + \varpi^{n-v}\mathcal{O}_{\tilde{M}(H\varpi^n)(w)}\right)$$

on $\tilde{M}(H\varpi^n)(w)$. $\mathcal{F}'_{n,v}$ is a $\mathcal{S}_{n,v}$–torsor, where the action is given by multiplication.

Definition 2.18. Let $\mathrm{Spf}(S)$ be an open affine of $\tilde{M}(H\varpi^n)(w)$ as constructed earlier, and let $x \in \mathcal{S}_{n,v}(\mathrm{Spf}(S))$. This can be rewritten in the form $x = ub$, where $u \in \mathcal{O}_{\mathcal{P}}^{\times}$ and $b \in (1 + \varpi^{n-v}S)$. Due to the condition on the valuation of the parameter $s \in K$, the exponential $b^s := \exp(s \log b)$ is well defined. Furthermore, $x^{\chi} := \chi(u)b^s$ does not depend on the choice of u and b. If $f \in \mathcal{O}_{\tilde{M}(H\varpi^n)(w)}(S)$ is a local section, then the $\mathcal{S}_{n,v}$**–action twisted by** χ is defined to be:

$$x \cdot f := x^{\chi} f$$

Definition 2.19 (The sheaf $\tilde{\Omega}_w^{\chi}$)**.** If χ is an n–accessible character, then the sheaf $\mathcal{O}^{\chi}_{\mathfrak{M}(H\varpi^n)(w)}$ is defined to be the original sheaf sheaf $\mathcal{O}_{\mathfrak{M}(H\varpi^n)(w)}$ but with $\mathcal{S}_{n,v}$–action twisted by χ. We define the sheaf $\tilde{\Omega}_w^{\chi}$ on $\tilde{M}(H\varpi^n)(w)$ by:

$$\tilde{\Omega}_w^{\chi} := \mathscr{H}om_{\mathcal{S}_{n,v}}(\mathcal{F}'_{n,v}, \mathcal{O}^{\chi^{-1}}_{\tilde{M}(H\varpi^n)(w)})$$

The subscript means that the sheaf $\tilde{\Omega}_w^\chi$ has an action of $\mathcal{S}_{n,v}$ too. The sheaf $\tilde{\Omega}_w^\chi$ is defined on $\tilde{M}(H\varpi^n)(w)$, and by going from the formal model to the rigid Shimura curves, we get a sheaf on $\mathfrak{M}(H\varpi^n)(w)\otimes_{F_\mathcal{P}} K$, again denoted $\tilde{\Omega}_w^\chi$.

Definition 2.20 (Overconvergent modular forms of weight χ and level $H \times K_1(\varpi^n)$)**.** Let L denote a complete, nonarchimedean field extension of $V[1/p]$. We define the space of *w–overconvergent modular forms of weight χ, tame level H, coefficients in L and with respect to the quaternion algebra D* as

$$S^D(L, H, \chi, w) := H^0(\mathfrak{M}(H\varpi^n)(w)\hat{\otimes}_{F_\mathcal{P}} L, \tilde{\Omega}_w^\chi).$$

If w' is another rational number such that $w' \geqslant w$ and $0 \leqslant w' < \frac{1}{q^{n-2}(q+1)}$, then the inclusion map $\mathfrak{M}(H\varpi^n)(w) \to \mathfrak{M}(H\varpi^n)(w')$ induces an inclusion $S^D(L, H, \chi, w') \hookrightarrow S^D(L, H, \chi, w)$. Therefore, one can also define

$$S^{\dagger, D}(H, \chi) := \lim_{w \to 0^+} H^0(\mathfrak{M}(H\varpi^n)(w)\hat{\otimes}_{F_\mathcal{P}} \mathbb{C}_p, \tilde{\Omega}_w^\chi)$$

as the space of *overconvergent modular forms of weight χ, tame level H, with coefficients in* $\mathbb{C}_p$.

2.3.4. *The sheaves $\tilde{\Omega}_{n,w}$.* Recall that there is an admissible affinoid covering $\{\mathcal{W}_n\}_{n \in \mathbb{N}}$ of the weight space $\mathcal{W}$. We are going to define sheaves whose global sections will be families of overconvergent modular forms over $\mathcal{W}_n$, for each $n \in \mathbb{N}$.

The two sheaves $\mathcal{F}'_{n,v}$ and $\mathcal{S}_{n,v}$ defined previously were defined on $\tilde{M}(H\varpi^n)(w)$. By going to the rigid Shimura curves, we obtain sheaves on $\mathfrak{M}(H\varpi^n)(w)$, and we'll denote these by $\mathcal{F}'_{n,v}$ and $\mathcal{S}_{n,v}$ again. Letting $\pi_1 : \mathcal{W}_n \times \mathfrak{M}(H\varpi^n) \to \mathcal{W}_n$ and $\pi_2 : \mathcal{W}_n \times \mathfrak{M}(H\varpi^n) \to \mathfrak{M}(H\varpi^n)$ be the projection maps we can consider

$\pi_2^{-1}(\mathcal{S}_{n,v})$ and $\pi_2^{-1}(\mathcal{F}'_{n,v})$ on $\mathcal{W}_n \times \mathfrak{M}(H\varpi^n)$, and we denote them by $\mathcal{S}_{n,v}$ and $\mathcal{F}'_{n,v}$ respectively again.

Let x be a section of $\mathcal{S}_{n,v}$. Write x as a product ub, where $u \in \mathcal{O}_{\mathcal{P}}^{\times}$ and $b \in 1 + \varpi^{n-v}\mathcal{O}_{\mathcal{W}_n \times \mathfrak{M}(H\varpi^n)(w)}$. Recall that if χ is an n–accessible character with parameters (i, s), then $b^s := \exp(s \log b)$ is well-defined, and the product $\chi(u)b^s$ does not depend on the choice of u and b. Thus, we can set $x^\chi := \chi(u)b^s$.

For a section $A\hat{\otimes}B$ of $\mathcal{O}_{\mathcal{W}_n \times \mathfrak{M}(H\varpi^n)(w)}$, define $x(A\hat{\otimes}B)$ to be the section of $\mathcal{O}_{\mathcal{W}_n \times \mathfrak{M}(H\varpi^n)(w)}$ corresponding to the function

$$(\chi, z) \mapsto x^\chi A(\chi)B(z).$$

Definition 2.21. We define the sheaf

$$\tilde{\Omega}_{n,w} := \mathscr{H}om_{\mathcal{S}_{n,v}}(\mathcal{F}'_{n,v}, \mathcal{O}_{\mathcal{W}_n \times \mathfrak{M}(H\varpi^n)(w)})$$

and the global sections of this sheaf forms the space of *families of w–overconvergent modular forms of tame level H over $\mathcal{W}_n$.*

2.4. **The choice of H in $K = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H$.** From now on, we will assume that the subgroup H appearing in the product $K = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H$ takes a specific form, namely the subgroup $H_1(N)$, defined below. We will be using the sets $S_{\mathbb{Q}}^{\lambda}$ and $S_E^{D,\lambda}$ that we defined in §2.1.1.

Let $\mathfrak{Q}$ be a prime of $\mathcal{O}_E$ such that $\mathfrak{Q} \in S_E^{D,\lambda}$. Assume that $\mathfrak{Q}$ lies above the prime $\mathfrak{q}$ in $\mathcal{O}_F$. Choose a uniformiser $\varpi_{\mathfrak{Q}}$ of $\mathcal{O}_{E_{\mathfrak{Q}}}$, and define

$$U_1(\mathfrak{Q}^r) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{E_{\mathfrak{Q}}}) \cong \mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{q}}}) : a \equiv 1 \bmod \varpi_{\mathfrak{Q}}^r, c \equiv 0 \bmod \varpi_{\mathfrak{Q}}^r \right\}$$

Now suppose that $N = \prod_{i=1}^{j} \mathfrak{Q}_i^{r_i}$ is a product of primes of $\mathcal{O}_E$ (different from $\mathcal{P}$), with all $\mathfrak{Q}_i \in S_E^{D,\lambda}$, and each $\mathfrak{Q}_i$ lying above the prime $\mathfrak{q}_i$ in $\mathcal{O}_F$. We will define the subgroup $H_1(N) \subset \Gamma$, by recalling that Γ decomposes as

$$(B \otimes_F F_{\mathcal{P}_2})^\times \times \cdots (B \otimes_F F_{\mathcal{P}_m})^\times \times \prod_{l \neq p} G(\mathbb{Q}_l)$$

and specifying the image of $H_1(N)$ in each of the groups $(B \otimes_F F_{\mathcal{P}_2})^\times, \cdots, (B \otimes_F F_{\mathcal{P}_m})^\times$ and $G(\mathbb{Q}_l)$ (for $l \neq p$) separately.

Let l be a rational prime.

(1) If $l \notin S_{\mathbb{Q}}^{\lambda}$, then none of the prime factors of N lie above l, and we take $H_1(N)_l := G(\mathbb{Z}_l)$.

(2) If $l \in S_{\mathbb{Q}}^{\lambda}$, then $G(\mathbb{Q}_l) \cong \mathbb{Q}_l^\times \times \prod_{\mathfrak{l}} (B \otimes_F F_{\mathfrak{l}})^\times$, with the product being over the primes of $\mathcal{O}_F$ over l. We will define the component $H_1(N)_{\mathfrak{l}}$ as follows:
   - if $\mathfrak{l} = \mathfrak{q}_i$ for some $\mathfrak{q}_i$, we define $H_1(N)_{\mathfrak{l}} := U_1(\mathfrak{Q}_i^{r_i}) \subset (B \otimes_F F_{\mathfrak{l}})^\times$
   - if $\mathfrak{l} \neq \mathfrak{q}_i$ for any $\mathfrak{q}_i$, set $H_1(N)_{\mathfrak{l}} := (\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{F_{\mathfrak{l}}})^\times$.

   If $l \neq p$, take $H_1(N)_l$ to be the product $\mathbb{Z}_l^\times \times \prod_{\mathfrak{l}} H_1(N)_{\mathfrak{l}}$. But if $l = p$, take $H_1(N)_p$ to be the product $H_1(N)_{\mathcal{P}_2} \times \ldots \times H_1(N)_{\mathcal{P}_m}$.

Finally, we define $H_1(N) := \prod_l H_1(N)_l$ to be the product over all the rational primes l.

2.5. **The Hecke operators and the classicality criterion.** On all the spaces of overconvergent modular forms, there are Hecke operators. There is a $U_{\mathcal{P}}$ operator, the $U_{\mathfrak{Q}}$ operators (for $\mathfrak{Q}|N$) and the $T_{\mathfrak{L}}$ operators (for $\mathfrak{L} \nmid \mathcal{P}N$). To simplify the notation, we will write $T_{\mathfrak{Q}}$ for the operator $U_{\mathfrak{Q}}$. We refer the reader to §2.3.3 of [Din14] and §5 of [Bra12] for their general definition, but we give one description of the $U_{\mathcal{P}}$ operator on fibres.

Consider a point of $\mathfrak{M}(H\varpi^n)(w)$, which consists of some data $(A, i, \theta, \overline{\alpha})$ as in Definition 2.1. Since w is certainly smaller than $\frac{q}{q+1}$, A has a canonical subgroup. For each subgroup D of $A[\varpi]_1^{2,1}$ which has trivial intersection with the canonical subgroup of A, let $t_2(D)$ denote the extension of D to a subgroup scheme of A. $t_2(D)$ has the property that in its $\mathcal{O}_D \otimes \mathbb{Z}_p$–decomposition, we have $t_2(D)_1^{2,1} = D$ and $t_2(D)_i^2 = A[q]_i^2$ for $2 \leqslant i \leqslant m$. With this, there is a way to take the quotient of $(A, i, \theta, \overline{\alpha})$ by $t_2(D)$ and get a tuple $(A/t_2(D), i', \theta', \overline{\alpha}')$ with its induced level structure, as done in §3.4 of [Kas04]. Furthermore, this tuple $(A/t_2(D), i', \theta', \overline{\alpha}')$ corresponds to a point in $\mathfrak{M}(H\varpi^n)(w/p)$, by Corollary 2.4 or [Kas09].

Then, there is a Hecke operator $U_{\mathcal{P}} : S^D(L, H, \chi, w) \to S^D(L, H, \chi, w)$, which has the following form on fibres: for a point $(A, i, \theta, \overline{\alpha})$ of $\mathfrak{M}(H\varpi^n)(w)$, we have

$$U_{\mathcal{P}}(f)(A, i, \theta, \overline{\alpha}) = \mathrm{Nm}_{F_{\mathcal{P}}/\mathbb{Q}_p}(\varpi) \sum_{D} pr^* f((A/t_2(C), i', \theta', \overline{\alpha}'))$$

where the sum is taken over all subgroups of $A[\varpi]_1^{2,1}$ that have trivial intersection with the canonical subgroup of A.

Definition 2.22. We define the Hecke algebra[1] $\mathcal{H}$ to be the free $\mathbb{Z}_p$–algebra generated by $U_{\mathcal{P}}$ and $T_{\mathfrak{Q}}$, as $\mathfrak{Q}$ ranges over all primes in $S_E^{D,\lambda}$.

2.5.1. *The classicality criterion.* Later on, we will need to use the the following classicality theorem, given as Theorem 5.1 in [Kas09].

Definition 2.23. Let L be a complete nonarchimedean field extension of $V[1/p]$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. For each rational number w with $0 \leqslant w < \frac{1}{q^{n-2}(q+1)}$, there is an

[1]Note that our Hecke algebra does not contain operators corresponding to all primes, and as such our eigencurve, to be defined, only classifies overconvergent systems of Hecke eigenvalues at all but a fixed finite set of primes.

inclusion $\mathfrak{M}(H\varpi^n)(w) \hookrightarrow \mathfrak{M}(H\varpi^n)$. This induces a map

$$S^D(L, H \times K_1(\varpi^n), k) \to S^{\dagger,D}(L, H \times K_1(\varpi^n), k).$$

The image of the above map is called the space of *classical forms of weight k and level $H \times K_1(\varpi^n)$ with coefficients in L.*

Recall that f is a generalised eigenform for $U_{\mathcal{P}}$ with eigenvalue $a_{\mathcal{P}}$ if we have $(U_{\mathcal{P}} - a_{\mathcal{P}})^N f = 0$ for some $N > 0$.

Theorem 2.24. Suppose $f \in S^{\dagger,D}(L, H \times K_1(\varpi^n), k)$, with L, k, n as above. If f is a generalised eigenform for $U_{\mathcal{P}}$ such that $v(a_{\mathcal{P}}) < k - v(\mathrm{Nm}_{F_{\mathcal{P}}/\mathbb{Q}_p}(\varpi))$, then f is classical.

3. The Hecke eigencurve $\mathcal{E}_D$

In [Buz07] Buzzard details the Hecke construction of the eigenvariety, a rigid analytic space formed from the following collection of global data:

- $\mathcal{W}$, a reduced rigid analytic space
- K, a complete field w.r.t a nontrivial, nonarchimedean norm $|.|_K$
- R, a reduced affinoid K–algebra
- $\mathbf{T}$ a commutative R–algebra
- a fixed element $\phi \in \mathbf{T}$.

as well as from the following local data:

- for each admissible affinoid open $X := \mathrm{Spm}(R_X) \subset \mathcal{W}$, there is a Banach R_X–module called M_X which satisfies condition (Pr) (that is to say, there is another Banach R_X–module N_X such that $M_X \oplus N_X$ is potentially orthonormalisable). See [Buz07, §2] for details.
- for each X as above, there is an R–module homomorphism $\mathbf{T} \to \mathrm{End}_{R_X}(M_X)$ such that the image of ϕ is always a compact operator on M_X
- for admissible open affinoids $Y \subset X \subset \mathcal{W}$ as above, there is a continuous homomorphism $\alpha : M_Y \to M_X \widehat{\otimes}_{R_X} R_Y$ which is a *link*, in the sense of [Buz07] (in the remarks preceding Lemma 5.6).

Denoting the eigenvariety associated to this data by $\mathcal{E}$, it possesses a natural projection $\pi : \mathcal{E} \to \mathcal{W}$, and the image of each irreducible component of $\mathcal{E}$ under π is always Zariski–dense in a component of $\mathcal{W}$.

In our situation, we will use

- the weight space from §2.2.4 for $\mathcal{W}$
- the Hecke algebra $\mathcal{H}$ defined in Definition 2.22 for $\mathbf{T}$
- the $U_{\mathcal{P}}$–operator for the element ϕ

- for $X := \mathrm{Spm}(R_X) \subset \mathcal{W}$ an affinoid of weight space, we know that $X \subset \mathcal{W}_n$ (for some $n \in \mathbb{N}$). By choosing w to be any positive rational smaller than $\frac{1}{q^{n-2}(q+1)}$, we define $M_X := \tilde{\Omega}_{n,w}(X)$. Recall that $\tilde{\Omega}_{n,w}$ is the sheaf defined in Definition 2.21.

It can be checked that our data satisfies all the conditions required for the construction of the eigenvariety, and we will write $\mathcal{E}_D$ for the resulting rigid space (the subscript is to show its origin from the quaternion algebra D used in §2.1.1). Furthermore, by Lemma 5.9 of of [Buz07], we have the following theorem. Recall that $S_\chi^{\dagger,D}$ denotes the space of overconvergent modular forms of weight χ and tame level H with coefficients in $\mathbb{C}_p$.

Theorem 3.1. Let $\pi : \mathcal{E}_D \to \mathcal{W}$ be the projection to weight space, and fix a weight $\chi \in \mathcal{W}(\mathbb{C}_p)$. Then there is a bijection between the subset $\pi^{-1}(\chi) \subset \mathcal{E}_D(\mathbb{C}_p)$ and the set of all systems of Hecke eigenvalues arising from eigenforms in $S_\chi^{\dagger,D}$ whose $U_{\mathcal{P}}$–eigenvalue is not zero.

3.1. **Family of Galois pseudo-representations over $\mathcal{E}_D$.** In the following we will show that to every system of Hecke eigenvalues (coming from a finite-slope overconvergent quaternionic eigenform of general weight), i.e. a point on $\mathcal{E}_D$, one can associate a Galois representation with the expected properties. We will then show that there is a family of pseudo-representations over $\mathcal{E}_D$ which interpolates these Galois representations.

3.2. **Pseudo-representations.**

Definition 3.2. Let G be a profinite group and R be a topological ring. An R–valued continuous pseudo-representation of dimension $d \in \mathbb{N}$ is a continuous function $T : G \to R$ such that

(1) $T(1) = d$

(2) $T(gh) = T(g)T(h)$ for all $g, h \in G$

(3) if S_{d+1} denotes the symmetric group on $d+1$ letters and $\epsilon : S_{d+1} \to \{\pm 1\}$ denotes the sign character, then

$$\sum_{\sigma \in S_{d+1}} \epsilon(\sigma) T_\sigma(g_1, g_2, \ldots, g_{d+1}) = 0$$

where, if $\sigma \in S_{d+1}$ has cycle decomposition

$$\sigma = \left(i_1^{(1)}, \ldots, i_{r_1}^{(1)}) \ldots (i_1^{(s)}, \ldots i_{r_s}^{(s)})\right)$$

then $T_\sigma : G^{d+1} \to R$ is the function

$$(g_1, g_2, \ldots, g_{d+1}) \mapsto T(g_{i_1^{(1)}} \ldots g_{i_{r_1}^{(1)}}) \ldots T(g_{i_1^{(s)}} \ldots g_{i_{r_s}^{(s)}}).$$

If $T : G \to R$ is a pseudo-representation, then we can extend it R–linearly to a pseudo-representation $T : R[G] \to R$, where the relations in (2) and (3) then hold for $g_1, g_2, \ldots, g_{d+1} \in R[G]$.

We have the following result about the relation between representations and pseudo-representations

Theorem 3.3. The following hold true.

(1) If $\rho : G \to \mathrm{GL}_d(R)$ is a representation, then $\mathrm{Trace}(\rho)$ is a pseudo-representation of dimension d.

(2) If R is an algebraically closed field of characteristic 0 and T is a pseudo-representation of dimension d, then there exists a unique semi–simple representation $\rho : G \to \mathrm{GL}_d(R)$ with $\mathrm{Trace}(\rho) = T$.

Recall from §2.1.1, the definitions of F, $E = F(\sqrt{\lambda})$, B, D, etc. Also recall our choice of tame level $H_1(N)$ from §2.4. Let S denote the set containing all the infinite places as well as the finite set of places of E dividing $\mathcal{P}N$ and the finite set of places of E not in $S_E^{D,\lambda}$. Write $G_{E,S}$ for the Galois group of the maximal algebraic extension of E which is unramified outside S. Recall that every prime $\mathfrak{l}$ of F induces two primes of E over it denoted $\mathfrak{L}, \bar{\mathfrak{L}}$ and $E_{\mathfrak{L}} \cong E_{\bar{\mathfrak{L}}} \cong F_{\mathfrak{l}}$. In particular, for every prime $\mathfrak{l}$ of F, we can restrict a Galois representation $\rho : G_{E,S} \to GL(V)$ to decomposition groups at $\mathfrak{L}$ and $\bar{\mathfrak{L}}$ obtaining two Galois representations $\rho_{\mathfrak{L}}, \rho_{\bar{\mathfrak{L}}} : G_{F_{\mathfrak{l}}} \to GL(V)$.

Given a classical quaternionic eigenform one can construct its associated p-adic Galois representation in the étale cohomology with $\overline{\mathbb{Q}}_p$ coefficients of the corresponding Shimura curve. More precisely, we have the following classical result due to Carayol. See [Car86b, Théorèm B'].

Theorem 3.4. Let $f \in S^D(\mathbb{C}_p, H_1(N) \times K_1(\varpi^n), k)$ be a classical quaternionic eigenform. Then, there exists a Galois representation

$$\rho_f : G_{E,S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that, for any $\mathfrak{L} \in S_E^{D,\lambda}$ that does not divide $\mathcal{P}N$, we have

$$\mathrm{Trace}(\rho(\mathrm{Frob}_{\mathfrak{L}})) = \text{ the } T_{\mathfrak{L}}\text{-eigenvalue of } f,$$

for a choice of geometric Frobenius $\mathrm{Frob}_{\mathfrak{L}}$ at $\mathfrak{L}$.

Definition 3.5. We say that ρ is a *tame level* $H_1(N)$ *modular Galois representation* if it is isomorphic to the Galois representation coming from some classical quaternionic eigenform f of tame level $H_1(N)$.

We can now use the theory of pseudo-representations to construct Galois representations attached to overconvergent quaternionic eigenforms of finite slope.

Proposition 3.6. There is a unique continuous pseudo-representation

$$\psi_{\mathcal{E}_D} : G_{E,S} \to \mathcal{O}(\mathcal{E}_D),$$

of dimension 2, such that at every classical point on $\mathcal{E}_D$, corresponding to an eigenform f, it specialises to the pseudo-representation ψ_f associated to ρ_f.

Proof. First we use [Che04, Proposition 6.4.6] along with Kassaei's classicality theorem 2.24, to show that the classical points are Zariski-dense on $\mathcal{E}_D$. The result then follows from [Che04, Proposition 7.1.1]. □

Corollary 3.7. Let f be a finite-slope overconvergent quaternionic eigenform of tame level $H_1(N)$. There is a Galois representation $\rho_f : G_{E,S} \to \mathrm{GL}_2(\mathbb{C}_p)$ such that, for any $\mathfrak{L} \in S_E^{D,\lambda}$ that does not divide $\mathcal{P}N$, we have $\mathrm{Trace}(\rho(\mathrm{Frob}_{\mathfrak{L}})) =$ the $T_{\mathfrak{L}}$-eigenvalue of f, for a choice of geometric Frobenius $\mathrm{Frob}_{\mathfrak{L}}$ at $\mathfrak{L}$.

Proof. For such f, let ψ_f denote the specialisation of $\psi_{\mathcal{E}_D}$ to the point corresponding to f. By part 2 of Theorem 3.3, we can find a Galois representation ρ_f whose associated pseudo-representation is ψ_f. The result now follows from continuity of Hecke eigenvalues on $\mathcal{E}_D$. □

4. The deformation-theoretic eigencurve $\mathcal{C}_D^{red}$

Following the ideas of Coleman and Mazur in [CM98], we construct a deformation–theoretic eigencurve $\mathcal{C}_D$ and prove that its nilreduction is isomorphic to $\mathcal{E}_D$.

Consider a tame level $H_1(N)$ modular Galois representation $\rho : G_{E,S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. We can find a stable lattice inside V, the underlying 2-dimensional $\overline{\mathbb{Q}}_p$-vector space on which $G_{E,S}$ acts. This lattice is a rank 2 $\mathcal{O}_L$–module, for some finite extension L of $\mathbb{Q}_p$. By restricting the $G_{E,S}$-action to this lattice, we obtain a representation $G_{E,S} \to \mathrm{GL}_2(\mathcal{O}_L)$. By reducing modulo the maximal ideal of $\mathcal{O}_L$, we get a mod p Galois representation $G_{E,S} \to \mathrm{GL}_2(k_L)$, where k_L is the residue field of L. The semisiplification of this representation is independent of the choice of the lattice and is denote by $\bar{\rho}$ and is called the *mod p reduction of ρ*.

Definition 4.1. Let $\overline{\rho}$ be a mod p Galois representation (that is, a Galois representation $\rho : G_{E,S} \to \mathrm{GL}_2(\mathbb{F})$ for some finite field $\mathbb{F}$). If $\overline{\rho}$ is isomorphic to the mod p reduction of some modular Galois representation of tame level $H_1(N)$, then we say that $\overline{\rho}$ is a *mod p modular Galois representation* of tame level $H_1(N)$.

Recall our running assumption $p \neq 2$. Fix a mod p modular Galois representation $\overline{\rho}$ of tame level $H_1(N)$ defined over some finite field $\mathbb{F}$ and let $\overline{\psi}$ be the pseudo-representation attached to $\overline{\rho}$. Since we're assuming that $p \neq 2$, there is a lifting of $\overline{\psi}$ to a universal pseudo-representation ψ^{univ}, which takes values in a complete noetherian local ring $R^{\mathrm{univ}}(\overline{\psi})$. We will also denote this ring by $R_{\overline{\rho}}$. Let $X_{\overline{\rho}} := (\mathrm{Spf}(R_{\overline{\rho}}))^{\mathrm{rig}}$. This is a rigid analytic space over $W(\mathbb{F})[1/p]$. In what follows, we will implicitly work with the base change of this rigid analytic space to $\mathbb{C}_p$, which will still be denoted $X_{\overline{\rho}}$. There is a natural map $\pi : X_{\overline{\rho}} \to \mathcal{W}$ such that for every point on $X_{\overline{\rho}}$ corresponding to a modular Galois representation ρ_f, the image under π is the weight of f considered as an overconvergent modular form. We then define $Y_{\overline{\rho}}$ as

$$Y_{\overline{\rho}} := X_{\overline{\rho}} \times \prod_{\mathfrak{L}|N} \mathbb{A}^1 \times \mathbb{G}_m$$

Here, the product is over all primes $\mathfrak{L} \in S_E^{D,\lambda}$ dividing N.

The objective now is to define a Zariski-closed subspace of $Y_{\overline{\rho}}$, which shall be the $\overline{\rho}$-component of our eigencurve.

Let $\mathcal{H}$ be the Hecke algebra defined in §2.22. By writing $x_{\mathfrak{L}}$ for the coordinate on $\mathbb{A}^1$, and $x_{\mathcal{P}}$ for the coordinate on $\mathbb{G}_m$, there is a map $\iota : \mathcal{H} \to \mathcal{O}(Y_{\overline{\rho}})$ which sends

$$T_{\mathfrak{L}} \mapsto \begin{cases} \mathrm{Trace}(\psi^{\mathrm{univ}}(\mathrm{Frob}_{\mathfrak{L}})) & \mathfrak{L} \nmid N \\ x_{\mathfrak{L}} & \mathfrak{L}|N \end{cases}$$

$$U_{\mathcal{P}} \mapsto x_{\mathcal{P}}^{-1}$$

We shall also define the subset $\tilde{\mathcal{H}} \subset \mathcal{H}$ to be the set of all $h \in \mathcal{H}$ such that $\iota(hU_{\mathcal{P}}) \in \mathcal{O}(Y_{\overline{\rho}})^{\times}$.

Choose a particular weight χ from the weight space. Recall that we have our space $S^{\dagger,D}(H_1(N), \chi)$ of overconvergent modular forms of weight χ and tame level $H_1(N)$, defined in the previous chapter. $\mathcal{H}$ acts on this space, and for $h \in \tilde{\mathcal{H}}$, we know that $hU_{\mathcal{P}}$ is a compact operator on $S^{\dagger,D}(H, \chi)$, which means it has a characteristic power series

$$P_{h,\chi}(T) := \det(1 - T \cdot hU_{\mathcal{P}}|_{S^{\dagger,D}(H,\chi)}).$$

Definition 4.2. We define $\mathcal{C}_{D,\overline{\rho}}$ to be the zero locus of $P_{h,\chi}(\iota(h)^{-1}x_{\mathcal{P}})$ on $Y_{\overline{\rho}}$ as $\chi \in \mathcal{W}$ and $h \in \tilde{\mathcal{H}}$ vary. We then define the global eigencurve $\mathcal{C}_D$ by taking the disjoint union of all the $\mathcal{C}_{D,\overline{\rho}}$, as $\overline{\rho}$ varies over all mod p tame level $H_1(N)$ modular Galois representations. We define $\mathcal{C}_D^{red}$ to be the nilreduction of $\mathcal{C}_D$.

We now have a deformation-theoretic eigencurve $\mathcal{C}_D$, and a Hecke eigencurve $\mathcal{E}_D$ (from the previous chapter). By Proposition 3.6, there is a family of pseudo-representations on $\mathcal{E}_D$ which over each component lifts a unique mod p modular Galois representation $\overline{\rho}$ of tame level $H_1(N)$. The universality of $X_{\overline{\rho}}$ gives a map $\mathcal{E}_D \to X_{\overline{\rho}}$. Using the analyticity of the Hecke eigenvalues on $\mathcal{E}_D$, we can extend this map to $Y_{\overline{\rho}}$. By Theorem 3.1, this map factors through $\mathcal{C}_D$. Since $\mathcal{E}_D$ is reduced, we obtain a rigid analytic map

$$j : \mathcal{E}_D \to \mathcal{C}_D^{red}.$$

The next goal is to prove the following theorem.

Theorem 4.3. The natural map $j : \mathcal{E}_D \to \mathcal{C}_D^{red}$ is an isomorphism of rigid analytic spaces.

The above is proven as Theorem 7.5.1 in [CM98], for the case of elliptic modular forms. The strategy of Coleman and Mazur is as follows: first, it is shown that the map $\mathcal{E}_D \to \mathcal{C}_D^{red}$ is a bijection on closed points. Next, it is shown that the map is a generic isomorphism (i.e., an isomorphism away from a finite number of points) and a local isomorphism (an isomorphism around every point). It is a general result in rigid analytic geometry [CM98, Lemma 7.5.2] that such a morphism is an isomorphism.

The proof that the map is a generic isomorphism as well as a local isomorphism follows from Coleman-Mazur's argument using general properties of eigenvarieties constructed via Buzzard's eigenvariety machine. So, for the remainder of this chapter, we will go through the proof that it is a bijection on points. If f is a simultaneous eigenform for $\mathcal{H}$, then we'll denote its system of Hecke eigenvalues by the map $\lambda_f : \mathcal{H} \to \mathbb{C}_p$ (in other words, for each $h \in \mathcal{H}$, $\lambda_f(h)$ is the h–eigenvalue of f).

It is clear from the description of j that, on points, j sends a system of Hecke eigenvalues $\lambda = \lambda_f$ (arising from some finite-slope overconvergent eigenform f) to the point $(\psi_f, (\lambda_f(T_{\mathfrak{L}}))_{\mathfrak{L}|N}, \lambda_f(U_{\mathcal{P}}))$ in $\mathcal{C}_D$. Here, ψ_f denotes the pseudo-representation attached to the Galois representation arising from f.

To prove that j is injective, let λ and λ' be two systems of Hecke eigenvalues satisfying $j(\lambda) = j(\lambda')$. We can immediately see that their $U_{\mathcal{P}}$–eigenvalues are the same, and also their $T_{\mathfrak{L}}$ eigenvalues are the same (for $\mathfrak{L}|N$). For primes $\mathfrak{L} \nmid \mathcal{P}N$, the trace of the pseudo-representation on $\mathrm{Frob}_{\mathfrak{L}}$ gives the $T_{\mathfrak{L}}$–eigenvalue. Since $j(\lambda) = j(\lambda')$, their associated pseudo-representations are the same, and hence we have $\lambda = \lambda'$.

For the proof that j is surjective, we split the argument into several steps. Let $\underline{c} = (\psi, (c_{\mathfrak{L}})_{\mathfrak{L}|N}, c_{\mathcal{P}}^{-1})$ be a point of $\mathcal{C}_D$, where ψ is a pseudo-representation, $c_{\mathfrak{L}} \in \mathcal{O}_{\mathbb{C}_p}$ for all $\mathfrak{L}|N$, and $c_{\mathcal{P}} \in \mathcal{O}_{\mathbb{C}_p}^{\times}$. Assume that the image of $\underline{c}$ under the projection $\mathcal{C}_D \to \mathcal{W}$ is the character χ. From our construction of $\mathcal{C}_D$, we know that

$$P_{h,\chi}(\iota(h)^{-1} c_{\mathcal{P}}^{-1}) = 0$$

for all $h \in \tilde{\mathcal{H}}$, so there is always a non–zero overconvergent modular form of weight χ which has $\iota(h) c_{\mathcal{P}}$ as its $hU_{\mathcal{P}}$–eigenvalue. We define the subspace $W_h \subset S_{\chi}^{\dagger,D}$ to be

$$W_h := \left\{ g \in S_\chi^{\dagger, D} : hU_{\mathcal{P}}(g) = \iota(h) c_{\mathcal{P}} g \right\}$$

which is always nonzero. In addition, as the Hecke algebra $\mathcal{H}$ is commutative, and maps W_h into itself, we can find a simultaneous basis of nonzero eigenforms for $\mathcal{H}$ in W_h.

Define the set

$$\mathcal{F}_h := \{\lambda_g : 0 \neq g \in W_h \text{ and } g \text{ is a simultaneous eigenform for all of } \mathcal{H}\}.$$

Since $W_h \neq \{0\}$, we also know that $\mathcal{F}_h \neq \varnothing$ is a finite set. Enumerate all the primes in the set $S_E^{D,\lambda}$ (different from $\mathcal{P}$) by $\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_n, \ldots$. In §4.0.1 we are going to define a subset $\tilde{\mathcal{H}}_0$ of $\tilde{\mathcal{H}}$, which depends on the point $\underline{c}$, and which contains 1 and satisfies the following property:
for each $i \in \mathbb{N}$, there is a $b_i \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the element

$$(1 + p^{b_1} T_{\mathfrak{L}_1}) \ldots (1 + p^{b_n} T_{\mathfrak{L}_n})$$

is in $\tilde{\mathcal{H}}_0$. Assume for now that we have constructed such a subset $\tilde{\mathcal{H}}_0$.

Proposition 4.4. If $\bigcap_{h \in \tilde{\mathcal{H}}_0} \mathcal{F}_h \neq \varnothing$, then $\underline{c}$ is in the image of $j : \mathcal{E}_D \to \mathcal{C}_D$.

Proof. Let $\lambda_f \in \bigcap_{h \in \tilde{\mathcal{H}}_0} \mathcal{F}_h$. It suffices to show that for all $n \geqslant 1$, $T_{\mathfrak{L}_n}(f) = \iota(T_{\mathfrak{L}_n}) f$.

Before beginning the proof, note that $1 \in \tilde{\mathcal{H}}_0$, so $f \in \mathcal{F}_1$ and hence $U_{\mathcal{P}} f = c_{\mathcal{P}} f$ by definition. We now proceed by induction on n.

Suppose $n = 1$. Then $(1 + p^{b_1} T_{\mathfrak{L}_1}) \in \tilde{\mathcal{H}}_0$, and so

$$(1 + p^{b_1} T_{\mathfrak{L}_1}) U_{\mathcal{P}}(f) = \iota(1 + p^{b_1} T_{\mathfrak{L}_1}) c_{\mathcal{P}} f$$

since $f \in \mathcal{F}_{(1+p^{b_1}T_{\mathfrak{L}_1})}$. But we can replace $U_{\mathcal{P}}f$ with $c_{\mathcal{P}}f$ on the left–hand side, and since $c_{\mathcal{P}} \neq 0$, this implies

$$(1 + p^{b_1}T_{\mathfrak{L}_1})(f) = \iota(1 + p^{b_1}T_{\mathfrak{L}_1})f$$

In addition to this, we know that ι is a ring homomorphism, hence

$$(1 + p^{b_1}T_{\mathfrak{L}_1})(f) = (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))(f)$$

Subtracting f from both sides and dividing by p^{b_1} gives us

$$T_{\mathfrak{L}_1}(f) = \iota(T_{\mathfrak{L}_1})(f)$$

as desired. For the inductive step, suppose that for $k = 1 \to n$, we have shown

$$T_{\mathfrak{L}_k}(f) = \iota(T_{\mathfrak{L}_k})(f).$$

By construction, the Hecke operator $(1 + p^{b_1}T_{\mathfrak{L}_1}) \dots (1 + p^{b_{n+1}}T_{\mathfrak{L}_{n+1}})$ is in $\tilde{\mathcal{H}}_0$. Therefore,

$$(1 + p^{b_1}T_{\mathfrak{L}_1}) \dots (1 + p^{b_{n+1}}T_{\mathfrak{L}_{n+1}})U_{\mathcal{P}}(f) = \iota(1 + p^{b_1}T_{\mathfrak{L}_1}) \dots \iota(1 + p^{b_{n+1}}T_{\mathfrak{L}_{n+1}})c_{\mathcal{P}}f$$

Again, we can replace $U_{\mathcal{P}}f$ with $c_{\mathcal{P}}f$, and divide both sides by $c_{\mathcal{P}}$. Furthermore, as $\mathcal{H}$ is commutative, the order of the brackets can be reversed to give

$$(*) \qquad (1 + p^{b_{n+1}}T_{\mathfrak{L}_{n+1}}) \dots (1 + p^{b_1}T_{\mathfrak{L}_1})(f) = \iota(1 + p^{b_{n+1}}T_{\mathfrak{L}_{n+1}}) \dots \iota(1 + p^{b_1}T_{\mathfrak{L}_1})(f)$$

By the inductive hypothesis, we know that

$$(1 + p^{b_n}T_{\mathfrak{L}_n}) \dots (1 + p^{b_1}T_{\mathfrak{L}_1})(f) = \iota(1 + p^{b_n}T_{\mathfrak{L}_n}) \dots \iota(1 + p^{b_1}T_{\mathfrak{L}_1})(f)$$

Substituting this expression into the left–hand side of $(*)$ and using the fact that $\iota(1+p^{b_i}T_{\mathfrak{L}_i})$ is a unit, we reduce $(*)$ to

$$(1+p^{b_{n+1}}T_{\mathfrak{L}_{n+1}})(f)=\iota(1+p^{b_{n+1}}T_{\mathfrak{L}_{n+1}})(f)$$

and hence we deduce $T_{\mathfrak{L}_{n+1}}(f)=\iota(T_{\mathfrak{L}_{n+1}})(f)$. This completes the proof by induction. □

To prove Proposition 4.4, we assumed the existence of a collection of Hecke operators $\tilde{\mathcal{H}}_0$ satisfying certain properties. We now prove the existence of such a collection by constructing it.

4.0.1. *The construction of $\tilde{\mathcal{H}}_0$: step 0.* We construct the set $\tilde{\mathcal{H}}_0$ inductively. Let χ denote the image of the point $\underline{c}$ under the map $\mathcal{C}_D\to\mathcal{W}$. Define $h_0:=1$ and put this into $\tilde{\mathcal{H}}_0$. Since $\iota(1)$ is a unit, $W_1\neq\phi$.

Definition 4.5. Define the set $\mathcal{S}$ to be the set of generalised eigenforms of $U_{\mathcal{P}}$ in $S_{\chi}^{\dagger,D}$ with $U_{\mathcal{P}}$–slope equal to $v(c_{\mathcal{P}})$ (here, v is the p–adic valuation on $\mathbb{C}_p$, normalised so that $v(p)=1$).

Clearly, $W_1\subset\mathcal{S}$, so $\mathcal{S}$ is not zero. On the other hand, the compactness of $U_{\mathcal{P}}$ on $S_{\chi}^{\dagger,D}$ tells us that $\mathcal{S}$ is finite–dimensional.

Lemma 4.6. If $g\in S_{\chi}^{\dagger,D}$ is such that $\lambda_g\in\mathcal{F}_h$, where $h=1+p\tau$ for some $\tau\in\mathcal{H}$, then $g\in\mathcal{S}$.

Proof. Since g is an eigenform for the whole of $\mathcal{H}$, there exist $a_{\mathcal{P}},a_\tau\in\mathcal{O}_{\mathbb{C}_p}$ such that

$$U_{\mathcal{P}}(g)=a_{\mathcal{P}}g \qquad \text{and} \qquad h(g)=(1+pa_\tau)(g).$$

Combining these tells us that $hU_{\mathcal{P}}(g) = a_{\mathcal{P}}(1 + pa_\tau)g$. Also, since $\lambda_g \in \mathcal{F}_h$, we know that $hU_{\mathcal{P}}(g) = \iota(h)c_{\mathcal{P}}$. Equating the eigenvalues of $hU_{\mathcal{P}}$, we have

$$a_{\mathcal{P}}(1 + pa_\tau) = \iota(h)c_{\mathcal{P}}$$

As $(1 + pa_\tau)$ and $\iota(h)$ are units, $a_{\mathcal{P}}$ and $c_{\mathcal{P}}$ have the same valuation, and hence $g \in \mathcal{S}$. □

Recall that $\mathcal{S}$ is finite–dimensional. Consequently, there are only finitely many eigenvalues for each Hecke operator on $\mathcal{S}$. One can choose $b_1 \in \mathbb{N}$ (large enough) such that if α, α' are eigenvalues for $U_{\mathcal{P}}$ on $\mathcal{S}$, then

$$v(\alpha - \alpha') \geqslant b_1 \Longrightarrow \alpha = \alpha'$$

4.0.2. *The construction of $\tilde{\mathcal{H}}_0$: step 1.* Define $h_1 := (1 + p^{b_1}T_{\mathfrak{L}_1})$, and put h_1 into $\tilde{\mathcal{H}}_0$. Now let $\lambda_{f_1} \in \mathcal{F}_{h_1}$. By the previous lemma, $f_1 \in \mathcal{S}$. Assume that

$$U_{\mathcal{P}}(f_1) = a_{\mathcal{P}}f_1 \qquad \text{and} \qquad T_{\mathfrak{L}_1}(f_1) = a_{\mathfrak{L}_1}f_1$$

where $a_{\mathcal{P}}, a_{\mathfrak{L}_1} \in \mathcal{O}_{\mathbb{C}_p}$. Combining the equations gives

$$(1 + p^{b_1}T_{\mathfrak{L}_1})U_{\mathcal{P}}(f_1) = (1 + p^{b_1}a_{\mathfrak{L}_1})a_{\mathcal{P}}f_1$$

On the other hand, $\lambda_{f_1} \in \mathcal{F}_{h_1}$, so

$$(1 + p^{b_1}T_{\mathfrak{L}_1})U_{\mathcal{P}}(f_1) = \iota(h_1)c_{\mathcal{P}}f_1$$

Equating the two expressions for the $h_1U_{\mathcal{P}}$–eigenvalue of f_1 gives

$$(1 + p^{b_1}a_{\mathfrak{L}_1})a_{\mathcal{P}} = \iota(h_1)c_{\mathcal{P}}$$

But $\iota(h_1) = (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))$, so

$$(1 + p^{b_1} a_{\mathfrak{L}_1})a_{\mathcal{P}} = (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}$$

This rearranges to give

$$(*) \qquad a_{\mathcal{P}} - c_{\mathcal{P}} = p^{b_1}(\iota(T_{\mathfrak{L}_1})c_{\mathcal{P}} - a_{\mathfrak{L}_1}a_{\mathcal{P}}).$$

If we now take the valuation of each side, then because the terms in $(\iota(T_{\mathfrak{L}_1})c_{\mathcal{P}} - a_{\mathfrak{L}_1}a_{\mathcal{P}})$ have non–negative valuation, we get

$$v(a_{\mathcal{P}} - c_{\mathcal{P}}) \geqslant b_1$$

However, since $a_{\mathcal{P}}$ and $c_{\mathcal{P}}$ are $U_{\mathcal{P}}$–eigenvalues, the above inequality implies that they must be equal. Hence, we have found that $f_1 \in \mathcal{F}_1$ (and $\mathcal{F}_{h_1} \subset \mathcal{F}_1$). We also note that upon substituting $a_{\mathcal{P}} = c_{\mathcal{P}}$ back into $(*)$, we obtain $\iota(T_{\mathfrak{L}_1}) = a_{\mathfrak{L}_1}$.

4.0.3. *The construction of $\tilde{\mathcal{H}}_0$: step 2.* Let $b_2 \in \mathbb{N}$ be such that if β and β' are $h_1U_{\mathcal{P}}$–eigenvalues on $\mathcal{S}$, then

$$v(\beta - \beta') \geqslant b_2 \Longrightarrow \beta = \beta'$$

Define $h_2 := (1 + p^{b_2}T_{\mathfrak{L}_2})h_1$. Pick any eigenform f_2 with the property that $\lambda_{f_2} \in \mathcal{F}_{h_2}$. Applying the previous lemma, we again have that $f_2 \in \mathcal{S}$. Since f_2 is an eigenform, there exist $a_{1,\mathcal{P}}$ and $a_{\mathfrak{L}_2} \in \mathcal{O}_{\mathbb{C}_p}$ such that

$$h_1U_{\mathcal{P}}(f_2) = a_{1,\mathcal{P}}f_2 \qquad \text{and} \qquad T_{\mathfrak{L}_2}(f_2) = a_{\mathfrak{L}_2}f_2$$

so upon combining these two, we get

$$h_2U_{\mathcal{P}}(f_2) = (1 + p^{b_2}a_{\mathfrak{L}_2})a_{1,\mathcal{P}}f_2$$

But, as $f_2 \in \mathcal{F}_{h_2}$, we also have

$$h_2 U_{\mathcal{P}}(f_2) = (1 + p^{b_2}\iota(T_{\mathfrak{L}_2}))(1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}} f_2$$

Equating the two eigenvalues of f_2 for $h_2 U_{\mathcal{P}}$ gives

$$(1 + p^{b_2} a_{\mathfrak{L}_2}) a_{1,\mathcal{P}} = (1 + p^{b_2}\iota(T_{\mathfrak{L}_2}))(1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}$$

and this rearranges to:

$$(**) \qquad a_{1,\mathcal{P}} - (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}} = p^{b_2}(a_{\mathfrak{L}_2} a_{1,\mathcal{P}} - \iota(T_{\mathfrak{L}_2})(1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}).$$

Now, $a_{1,\mathcal{P}}$ is a $h_1 U_{\mathcal{P}}$–eigenvalue on $\mathcal{S}$ (because it arises from f_2). Also, $(1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}$ is the same as $\iota(h_1)c_{\mathcal{P}}$, which is also a $h_1 U_{\mathcal{P}}$–eigenvalue on $\mathcal{S}$ (it arises from f_1). Hence, by taking the valuation of both sides, the factor of p^{b_2} tells us

$$v(a_{1,\mathcal{P}} - (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}) \geqslant b_2$$

and so $a_{1,\mathcal{P}} = (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}$, which equals $\iota(h_1)c_{\mathcal{P}}$. From this, we deduce that $\lambda_{f_2} \in \mathcal{F}_{h_1}$, and that $\mathcal{F}_{h_2} \subset \mathcal{F}_{h_1}$. Also, plugging in $a_{1,\mathcal{P}} = (1 + p^{b_1}\iota(T_{\mathfrak{L}_1}))c_{\mathcal{P}}$ into $(**)$ also tells us that $a_{\mathfrak{L}_2} = \iota(T_{\mathfrak{L}_2})$.

We continue in this way, to construct the set $\tilde{\mathcal{H}}_0$ as $\tilde{\mathcal{H}}_0 = \{h_0, h_1, h_2, h_3, \ldots\}$. The h_i are such that

$$\mathcal{F}_{h_0} \supset \mathcal{F}_{h_1} \supset \mathcal{F}_{h_2} \supset \mathcal{F}_{h_3} \ldots$$

Since each $\mathcal{F}_{h_i}$ is finite, and $\bigcap_{i=0}^{n} \mathcal{F}_{h_i} \neq \varnothing$ for any $n \geqslant 0$, it is immediate to see that we have

$$\bigcap_{i=0}^{\infty} \mathcal{F}_{h_i} \neq \varnothing.$$

This proves that the point $\underline{c}$ is in the image of j, because by Proposition 4.4, j maps any element of $\bigcap_{i=0}^{\infty} \mathcal{F}_{h_i}$ to $\underline{c}$.

5. p-adic Hodge theory of overconvergent quaternionic automorphic forms

5.1. **Background from p-adic Hodge theory.**

5.1.1. *The fields and the Galois groups.* Let k be a perfect field of characteristic p (so that the map $x \mapsto x^p$ is an automorphism of k). Let $F := W(k)[1/p]$, where $W(k)$ denotes the ring of Witt vectors of k. Then the ring of integers of F is $\mathcal{O}_F = W(k)$. We also assume that K is a totally ramified finite extension of F, and write $\mathbf{C} := \widehat{\overline{K}}$ for the p–adic completion of an algebraic closure of K. The typical case is when k is a finite extension of $\mathbb{F}_p$, and K is a finite extension of $\mathbb{Q}_p$, whence $\mathbf{C} = \mathbb{C}_p$, the p–adic complex numbers.

Now, let μ_m be the group of m–th roots of unit in $\overline{K}$, and $\varepsilon = (\varepsilon^{(n)})_{n \geqslant 0}$ be a compatible sequence of p^n–th roots of unity in $\overline{K}$, i.e.

$$\varepsilon^{(0)} = 1, \ \varepsilon^{(1)} \neq 1, \text{ and } (\varepsilon^{(n+1)})^p = \varepsilon^{(n)} \text{ for } n \geqslant 0.$$

We also write $K_n := K(\varepsilon^{(n)})$ and $K_\infty := \bigcup_{n=0}^{\infty} K_n$ and similarly for F. Thus, we have a tower of fields

$$F \subset K \subset K_n \subset K_\infty \subset \overline{F} = \overline{K} \subset \mathbf{C}$$

with Galois groups $G_K := \mathrm{Gal}(\overline{K}/K)$, $H_K := \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K := \mathrm{Gal}(K_\infty/K)$.

Recall that we have the usual cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$. This is given as follows: if $\sigma \in G_K$, then there exist $a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ such that

$$\sigma(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{a_n}$$

and $(\dots, a_n, \dots, a_2, a_1)$ is a well–defined element of $\mathbb{Z}_p^\times$. We then define $\chi(\sigma) = (\dots, a_n, \dots, a_2, a_1)$. Furthermore, H_K is the kernel of χ, so we can identfiy Γ_K with an open subgroup of $\mathbb{Z}_p^\times$.

The Ax–Sen–Tate theorem says that if $H \subset G_K$ is a closed subgroup, then $\widehat{\overline{K}^H} = \mathbf{C}^H$.

5.1.2. *The field* $\mathbf{B_{dR}}$ *and the ring* $\mathbf{B_{cris}}$. Let $\mathbf{C} = \mathbb{C}_p$ with valuation val_p normalised so that $\mathrm{val}_p(p) = 1$, and let $\tilde{\mathbf{E}}^+$ be the set

$$\tilde{\mathbf{E}}^+ = \lim_{x \mapsto x^p} \mathcal{O}_{\mathbf{C}} = \left\{(x^{(0)}, x^{(1)}, \dots) \middle| (x^{(i+1)})^p = x^{(i)} \ \forall i \geqslant 0\right\}$$

Define addition and multiplication on elements $x = (x^{(i)}), y = (y^{(i)})$ in $\tilde{\mathbf{E}}^+$ as follows:

$$(x+y)^{(i)} = \lim_{j \to \infty} (x^{(i+j)} + y^{(i+j)})^{p^j}$$
$$(xy)^{(i)} = x^{(i)} y^{(i)}$$

This makes $\tilde{\mathbf{E}}^+$ into a perfect local ring of characteristic p, and we define $\tilde{\mathbf{E}}$ to be its fraction field. $\tilde{\mathbf{E}}^+$ is equipped with a valuation $v_{\tilde{\mathbf{E}}^+}(x) = \mathrm{val}_p(x^{(0)})$ which naturally extends to a valuation $v_{\tilde{\mathbf{E}}}$ on $\tilde{\mathbf{E}}$. For example, if $\varepsilon = (\varepsilon^{(i)})$ is as previously defined, then

$$v_{\tilde{\mathbf{E}}^+}(\varepsilon - 1) = \mathrm{val}_p\left(\lim_{j \to \infty} (\varepsilon^{(j)} - 1)^{p^j}\right) = \frac{p}{p-1}$$

Now, define $\tilde{\mathbf{A}}^+ := W(\tilde{\mathbf{E}}^+)$. Being the ring of Witt vectors of $\tilde{\mathbf{E}}^+$, there is the Teichmüller map $[\cdot] : \tilde{\mathbf{E}}^+ \to \tilde{\mathbf{A}}^+$ which is a section of the canonical map $\tilde{\mathbf{A}}^+ \to \tilde{\mathbf{E}}^+$, and every element $x \in \tilde{\mathbf{A}}^+$ can be uniquely written in the form

$$x = \sum_{k=0}^{\infty} p^k [x_k] \quad \text{where } x_k \in \tilde{\mathbf{E}}^+.$$

We also define $\tilde{\mathbf{B}}^+$, to be the fraction field of $\tilde{\mathbf{A}}^+$. Its elements are series of the form

$$\sum_{k \gg -\infty}^{\infty} p^k [x_k] \quad \text{where } x_k \in \tilde{\mathbf{E}}^+.$$

There is also a natural map $\theta : \tilde{\mathbf{B}}^+ \to \mathbf{C}$, such that

$$\theta\left(\sum_{k \gg -\infty}^{\infty} p^k [x_k]\right) = \sum_{k \gg -\infty}^{\infty} p^k x_k^{(0)}$$

which is in fact surjective. If we let $\varepsilon_1 \in \tilde{\mathbf{E}}^+$ be the sequence $(\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots)$, then $\varepsilon_1^p = \varepsilon$ and so the element

$$\omega := \frac{[\varepsilon] - 1}{[\varepsilon_1] - 1}$$

is an element of $\tilde{\mathbf{B}}^+$ which is in the kernel of θ. In fact, it can be shown that $\mathrm{Ker}(\theta)$ is the ideal generated by ω, which leads to the definition of the rings $\mathbf{B}_{\mathbf{dR}}^+$ and $\mathbf{B}_{\mathbf{dR}}$.

Definition 5.1. Define $\mathbf{B}_{\mathbf{dR}}^+$ to be the completion of $\tilde{\mathbf{B}}^+$ w.r.t the $\mathrm{Ker}\theta$–adic topology. Thus,

$$\mathbf{B}_{\mathbf{dR}}^+ := \varprojlim_n \tilde{\mathbf{B}}^+/(\mathrm{Ker}\theta)^n$$

and every element of $\mathbf{B}_{\mathbf{dR}}^+$ can be written (non–uniquely!) in the form

$$x = \sum_{n=0}^{\infty} x_n \omega^n$$

where the $x_n \in \tilde{\mathbf{B}}^+$. Additionally, since $\theta(1 - [\varepsilon]) = 0$ (because $\varepsilon^{(0)} = 1$), the following series

$$t := -\sum_{n=1}^{\infty} \frac{(1 - [\varepsilon])^n}{n}$$

converges in $\mathbf{B}_{\mathbf{dR}}^+$. The series can be thought of as the Taylor expansion of $\log(1-(1-[\varepsilon]))$, or rather $\log([\varepsilon])$. We then define $\mathbf{B}_{\mathbf{dR}}$ to be $\mathbf{B}_{\mathbf{dR}}^+[1/t]$, which is a field.

Definition 5.2. Define the ring $\mathbf{B}_{\mathbf{cris}}^+$ to be the following subring of $\mathbf{B}_{\mathbf{dR}}^+$:

$$\mathbf{B}_{\mathbf{cris}}^+ := \left\{ x \in \mathbf{B}_{\mathbf{dR}}^+ \middle| x \text{ can be written in the form } \sum_{n=0}^{\infty} x_n \frac{\omega^n}{n!} \text{ where } x_n \to 0 \text{ in } \tilde{\mathbf{B}}^+ \right\}$$

We also define $\mathbf{B}_{\mathbf{cris}} := \mathbf{B}_{\mathbf{cris}}^+[1/t]$. However, unlike $\mathbf{B}_{\mathbf{dR}}$, $\mathbf{B}_{\mathbf{cris}}$ is not a field.

Since the Galois group G_K acts on $\mathcal{O}_{\mathbf{C}}$, we get a Galois action on each of the rings $\tilde{\mathbf{E}}^+$, $\tilde{\mathbf{A}}^+$, $\tilde{\mathbf{B}}^+$ and also on $\mathbf{B}_{\mathbf{dR}}^+$, $\mathbf{B}_{\mathbf{dR}}$, $\mathbf{B}_{\mathbf{cris}}^+$ and $\mathbf{B}_{\mathbf{cris}}$. For example, if $g \in G_K$, then $g \cdot t = \chi(g)t$ acts on t via the cyclotomic character. In fact, $(\mathbf{B}_{\mathbf{dR}})^{G_K} = K$ and $(\mathbf{B}_{\mathbf{cris}})^{G_K}$ is the maximal absolutely unramified subfield of K.

We can thus define the functors $\mathbf{D}_{\mathbf{cris}}^+$ and $\mathbf{D}_{\mathbf{dR}}^+$. Let V be a G_K–representation. Define

$$\mathbf{D}_{\mathbf{cris}}^+(V) := (\mathbf{B}_{\mathbf{cris}}^+ \widehat{\otimes}_K V)^{G_K}$$

$$\mathbf{D}_{\mathbf{dR}}^+(V) := (\mathbf{B}_{\mathbf{dR}}^+ \widehat{\otimes}_K V)^{G_K}.$$

These will not be the spaces we use in the proof of the properness of the eigencurve however. For dimensional reasons, we will need to cut out a direct summand, and this process will be made more explicit later on.

Note that our original ring $\tilde{\mathbf{E}}^+$ comes with a Frobenius map. This extends to a Frobenius map on $\tilde{\mathbf{A}}^+$, on $\tilde{\mathbf{B}}^+$, and then onto $\mathbf{B}_{\mathbf{cris}}^+$ and $\mathbf{D}_{\mathbf{cris}}^+(V)$. By Theorem 9.1.8 of [BC09], the Frobenius map on $\mathbf{B}_{\mathbf{cris}}^+$ is injective, hence it is also injective on $\mathbf{D}_{\mathbf{cris}}^+(V)$ because V is finite–dimensional.

5.1.3. *The rings* $\mathbf{B}, \mathbf{B}^{\dagger,s}, \mathbf{B}_K^{\dagger,s}, \mathbf{B}_{rig}^{\dagger,s}$ *and* $\mathbf{B}_{rig,s}^{\dagger,s}$. Let π_K be a formal variable, and let F' denote the maximal unramified extension of F in K_∞. Consider the ring

$$\mathbf{A}_K := \left\{ \sum_{k=-\infty}^{\infty} a_k \pi_K^k : a_k \in O_{F'}, a_{-k} \to 0 \text{ as } k \to \infty \right\}$$

Note that $\mathbf{A}_K/p = k_{F'}((\pi_K))$, where $k_{F'}$ us the residue field of $O_{F'}$, and that $\mathbf{A}_K$ is a Cohen ring. We'll write $\mathbf{E}_K$ for $\mathbf{A}_K/p$.

The p–th power map on the field $\mathbf{E}_K$ extends to a Frobenius map φ on $\mathbf{A}_K$, which satisfies $\varphi(\pi_K) = \pi_K^p (\mathrm{mod} p)$. There is also an action of Γ_K on $\mathbf{E}_K$ (hence on $\mathbf{A}_K$), but the formula describing this action depends on K. In the case where $K = F$, Frobenius and Galois have simple descriptions:

$$\varphi(\pi_K) = (1 + \pi_K)^p - 1$$

$$\gamma(\pi_K) = (1 + \pi_K)^{\chi(\gamma)} - 1$$

There is also a way to embed $\mathbf{E}_K$ into $\tilde{\mathbf{E}}$, but again this depends on what K is; in the simplest case $K = F$, the map $\mathbf{E}_F = k_F((\pi_F)) \to \tilde{\mathbf{E}}$ sends π_F to $\varepsilon - 1$. Write $\mathbf{E}$ for the completion of the separable closure of $\mathbf{E}_K$ in $\tilde{\mathbf{E}}$, and set $G_{\mathbf{E}_K} := \mathrm{Gal}(\mathbf{E}/\mathbf{E}_K)$.

Now let $\mathbf{B}_K$ be the fraction field of $\mathbf{A}_K$ (thus, $\mathbf{B}_K = \mathbf{A}_k[1/p]$). Set $\tilde{\mathbf{A}} := W(\tilde{\mathbf{E}})$ and take $\tilde{\mathbf{B}}$ to be the fraction field of $\tilde{\mathbf{A}}$ (so $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[1/p]$). The embedding of $\mathbf{E}_K$ into $\tilde{\mathbf{E}}$ induces an inclusion of $\mathbf{A}_K$ into $\tilde{\mathbf{A}}$ and hence a map $\mathbf{B}_K \hookrightarrow \tilde{\mathbf{B}}$. This makes $\mathbf{B}_K$ into a subfield of $\tilde{\mathbf{B}}$.

Write $\mathbf{B}$ for the completion of the maximal unramified extension of $\mathbf{B}_K$ in $\tilde{\mathbf{B}}$. Now set $\mathbf{A} := \mathbf{B} \cap \tilde{\mathbf{A}}$. The Galois group $G_{\mathbf{E}_K}$ acts on $\mathbf{E}_K$, hence also on $\mathbf{A}_K$ and $\mathbf{B}_K$. Therefore it also acts on $\mathbf{B}$, and it can be shown that $\mathbf{B}^{G_{\mathbf{E}_K}} = \mathbf{B}_K$.

From its definition, every element of $\tilde{\mathbf{B}}$ can be written in the form

$$\sum_{k \gg -\infty}^{\infty} p^k [x_k]$$

where the $x_k \in \tilde{\mathbf{E}}$. So, for $s \in \mathbb{R}_{>0}$, define $\mathbf{B}^{\dagger,s}$ to be the following subring of $\mathbf{B}$:

$$\mathbf{B}^{\dagger,s} := \left\{ x \in \mathbf{B} : x = \sum_{k \gg -\infty}^{\infty} p^k [x_k], \text{ and } k + \frac{p-1}{ps} v_{\mathbf{E}}(x_k) \to +\infty \text{ as } k \to +\infty \right\}$$

When $s = p^{n-1}(p-1)$ for some $n \geqslant 0$, the condition on the valuation is equivalent to requiring that the element $\sum_{k \gg -\infty}^{\infty} p^k [x_k^{p^{-n}}]$ converges in $\mathbf{B}_{\mathbf{dR}}^+$, or that $\sum_{k \gg -\infty}^{\infty} p^k x_k^{(n)}$ converges in $\mathbf{C}$.

By taking H_K–invariants (remembering that $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$), one obtains a ring denoted $\mathbf{B}_K^{\dagger,s}$ which has the following description:

$$\mathbf{B}_K^{\dagger,s} := \left\{ f(\pi_K) = \sum_{k \gg -\infty}^{\infty} a_k \pi_K^k : a_k \in F' \text{ and } f(X) \text{ is conv. \& bdd on } p^{-1/se_K} \leqslant |X| < 1 \right\}.$$

Here, e_K is the ramification index of the extension K_∞/F_∞ (and which is not necessarily $[K_\infty : F_\infty]$). We then let $\mathbf{B}_K^{\dagger} := \bigcup_{s>>0} \mathbf{B}_K^{\dagger,s}$.

By dropping the boundedness condition, there are similar definitions for the rings $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ and $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$. We first define $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ as

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} := \left\{ f(\pi_K) = \sum_{k \gg -\infty}^{\infty} a_k \pi_K^k : a_k \in F' \text{ and } f(X) \text{ is convergent on } p^{-1/se_K} \leqslant |X| < 1 \right\}.$$

and then take the union $\mathbf{B}_{\mathrm{rig},K}^{\dagger} := \bigcup_{s>>0} \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$.

5.1.4. *The functors $\boldsymbol{D}_*(\cdot)$ for families of Galois representations over an affinoid space.* In Appendix B.6 and §4 of [Bel13], Bellovin shows how to construct sheafified versions of the functors $\mathbf{D}_*(\cdot)$, which can be applied to a family of Galois representations V_X over X. The resulting sheaf is coherent, and it is proven in §4.2 of [Bel13] that some of the theorems comparing these p–adic Hodge invariants $\mathbf{D}_*(\cdot)$ with certain (φ, Γ)–modules also hold in the case of families of Galois representations.

Since our family of Galois representations will only be taken over affinoid spaces $X = \mathrm{Sp}(S)$, the situation is simpler, so we will go ahead and define the functors $\mathbf{D}_*(V_X)$ that we need.

Definition 5.3. Let $X = \mathrm{Sp}(R)$ be an affinoid space and V_X be a family of G_K–representations. Define

$$\mathbf{D}^+_{\mathbf{dR}}(V_X) := ((R \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^+_{\mathbf{dR}}) \widehat{\otimes}_R V_X)^{G_K}$$

$$\mathbf{D}^+_{\mathbf{cris}}(V_X) := ((R \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^+_{\mathbf{cris}}) \widehat{\otimes}_R V_X)^{G_K}$$

The Frobenius map on $\mathbf{B}^+_{\mathbf{cris}}$ (recall that this is injective) extends to a Frobenius map on $\mathbf{D}^+_{\mathbf{cris}}(V_X)$ which is also injective.

We will also need the definitions of the modules $\mathbf{D}^+_{\mathbf{dif}}(V_X)$ and $\mathbf{D}^\dagger_{\mathbf{rig}}(V_X)$. These are defined in §3.2 of [DL14] for $G_{\mathbb{Q}_p}$ representations, and more generally in [KL10] for families of G_K–representations.

5.1.5. *The σ–part of the decomposition of a module.* As mentioned before, we will not be using the full spaces $\mathbf{D}^+_{\mathbf{dR}}(V_X)$, $\mathbf{D}^+_{\mathbf{cris}}(V_X)$ etc. Instead, we will decompose them and focus on only one of the direct summands, which we will informally refer to as "the σ–part".

Recall f, E and D from §2.1.1. For $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$–modules M, we saw that there was a decomposition

$$M \xrightarrow{\sim} M_1^1 \oplus \cdots \oplus M_m^1 \oplus M_1^2 \oplus \cdots \oplus M_m^2$$

with M_1^2 further decomposing into $M_1^{2,1} \oplus M_1^{2,2}$.

Denote the real embeddings of F by $\sigma_i : F \hookrightarrow \mathbb{R}$, for $1 \leqslant i \leqslant d$. Let $\mathbb{R} \hookrightarrow \mathbb{C}$ denote the canonical injection, and fix a choice of isomorphism $\mathbb{C} \cong \mathbb{C}_p$. By composing each σ_i with these two maps, we get a collection of embeddings of $F \hookrightarrow \mathbb{C}_p$. Write σ_i again for these embeddings. In particular, each embedding factors through $\overline{\mathbb{Q}}_p$, so write $\sigma_i : F \hookrightarrow \overline{\mathbb{Q}}_p$ for these embeddings. Note that there is an isomorphism

$$F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \bigoplus_{\sigma_i} \overline{\mathbb{Q}}_p$$

given by

$$a \otimes e \mapsto (\sigma_1(a)e, \sigma_2(a)e, \ldots, \sigma_d(a)e)$$

What this means is that any $F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module M decomposes as the direct sum of $\overline{\mathbb{Q}}_p$–modules

$$(\dagger) \qquad M_{\sigma_1} \oplus M_{\sigma_2} \oplus \ldots \oplus M_{\sigma_d}$$

Since $E = F(\sqrt{\lambda})$ and $\mu \in \mathbb{Q}_p$ is a squareroot of λ, each embedding $\sigma_i : F \hookrightarrow \overline{\mathbb{Q}}_p$ gives rise to two embeddings $E \hookrightarrow \overline{\mathbb{Q}}_p$. For $x, y \in F$ the two embeddings are

$$x + y\sqrt{\lambda} \mapsto \sigma_i(x) + \mu\sigma_i(y)$$

$$x + y\sqrt{\lambda} \mapsto \sigma_i(x) - \mu\sigma_i(y)$$

By abuse of notation, denote the first map by σ_i again and the second map by $\overline{\sigma}_i$. These induce an isomorphism

$$E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \left(\bigoplus_{\sigma_i} \overline{\mathbb{Q}}_p\right) \oplus \left(\bigoplus_{\overline{\sigma}_i} \overline{\mathbb{Q}}_p\right)$$

given by

$$a \otimes e \mapsto \Big(\sigma_1(a)e, \sigma_2(a)e, \ldots, \sigma_d(a)e\Big) \oplus \Big(\overline{\sigma}_1(a)e, \overline{\sigma}_2(a)e, \ldots, \overline{\sigma}_d(a)e\Big)$$

So, any $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module N decomposes as a sum of $\overline{\mathbb{Q}}_p$–modules

$$N_{\sigma_1} \oplus N_{\sigma_2} \oplus \ldots \oplus N_{\sigma_d} \oplus N_{\overline{\sigma}_1} \oplus N_{\overline{\sigma}_2} \oplus \ldots \oplus N_{\overline{\sigma}_d}$$

Let S be the base scheme of the universal abelian variety in the moduli problem 2.1 defined over E. If M is the Lie algebra of this universal abelian variety, then M has the structure of a $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$–module and condition (1) of the moduli problem tells us $M_1^{2,1}$ is a projective $\mathcal{O}_S$–module of rank 1.

By writing $S_{\overline{\mathbb{Q}}_p}$ for the base change of S to $\overline{\mathbb{Q}}_p$, we can base change the universal abelian variety to $S_{\overline{\mathbb{Q}}_p}$. Its resulting Lie Algebra is $M_{\overline{\mathbb{Q}}_p} := M \otimes \overline{\mathbb{Q}}_p$, which is a $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module, and so decomposes as

$$M_{\overline{\mathbb{Q}}_p,\sigma_1} \oplus M_{\overline{\mathbb{Q}}_p,\sigma_2} \oplus \ldots \oplus M_{\overline{\mathbb{Q}}_p,\sigma_d} \oplus M_{\overline{\mathbb{Q}}_p,\overline{\sigma}_1} \oplus M_{\overline{\mathbb{Q}}_p,\overline{\sigma}_2} \oplus \ldots \oplus M_{\overline{\mathbb{Q}}_p,\overline{\sigma}_d}$$

However, $M_1^{2,1} \otimes \overline{\mathbb{Q}}_p$ is a submodule of $M_{\overline{\mathbb{Q}}_p}$ and has rank 1 as an $\mathcal{O}_{S_{\overline{\mathbb{Q}}_p}}$–module. Therefore, it must be contained in exactly one of the $M_{\overline{\mathbb{Q}}_p,\sigma_i}$ or $M_{\overline{\mathbb{Q}}_p,\overline{\sigma}_i}$, and we let $\sigma := \sigma_i$ denote the component it lies in.

We will now focus on the case where the field $K = F_{\mathcal{P}}$, and $X = \mathrm{Sp}(R)$ is an affinoid (where R is a $\overline{\mathbb{Q}}_p$–algebra), and define the modified functors $\mathbf{D}^+_{\mathbf{dR},\sigma}(V_X)$, $\mathbf{D}^+_{\mathbf{cris},\sigma}(V_X)$, $\mathbf{D}^+_{\mathbf{dif},\sigma}(V_X)$ and $\mathbf{D}^\dagger_{\mathbf{rig},\sigma}(V_X)$.

Consider $\mathbf{D}^+_{\mathbf{dR}}(V_X)$. Since $(\mathbf{B}^+_{\mathbf{dR}})^{G_{F_{\mathcal{P}}}} = F_{\mathcal{P}}$, we know that $\mathbf{D}^+_{\mathbf{dR}}(V_X)$ has the structure of a $F_{\mathcal{P}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module. We define $\mathbf{D}^+_{\mathbf{dR},\sigma}(V_X)$ to be the σ–part of $\mathbf{D}^+_{\mathbf{dR}}(V_X)$

Now look at $\mathbf{D}^+_{\mathbf{cris}}(V_X)$. Since $(\mathbf{B}^+_{\mathbf{cris}})^{G_{F_{\mathcal{P}}}} = F_{\mathcal{P},0}$ (the maximal unramified extension of $\mathbb{Q}_p$ inside $F_{\mathcal{P}}$) we only have that $\mathbf{D}^+_{\mathbf{cris}}(V_X)$ is a $F_{\mathcal{P},0} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module. Therefore, we need to look at $\mathbf{D}^+_{\mathbf{cris}}(V_X) \otimes_{F_{\mathcal{P},0}} F_{\mathcal{P}}$, which is a $F_{\mathcal{P}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module. Define $\mathbf{D}^+_{\mathbf{cris},\sigma}(V_X)$ to be the σ–part of $\mathbf{D}^+_{\mathbf{cris},\sigma}(V_X) \otimes_{F_{\mathcal{P},0}} F_{\mathcal{P}}$.

Similarly, for $\mathbf{D}^\dagger_{\mathbf{rig}}(V_X)$ and $\mathbf{D}^+_{\mathbf{dif}}(V_X)$ we can define their σ-part $\mathbf{D}^\dagger_{\mathbf{rig},\sigma}(V_X)$ and $\mathbf{D}^+_{\mathbf{dif},\sigma}(V_X)$ respectively.

Proposition 5.4. Let $X = \mathrm{Sp}(R)$ where R is $\mathbb{C}_p$–affinoid algebra and V_X be an R-linear representation of $G_{F_{\mathcal{P}}}$. We have $\mathbf{D}^+_{\mathbf{dR},\sigma}(V_X) \cong \mathbf{D}^+_{\mathbf{dif},\sigma}(V_X)^\Gamma$ and $\mathbf{D}^+_{\mathbf{cris},\sigma}(V_X) \cong \mathbf{D}^\dagger_{\mathbf{rig},\sigma}(V_X)^\Gamma$.

Proof. This follows from taking the σ-component in the isomorphisms given by Theorem 4.2.7 and Theorem 4.2.8 of [Bel13]. □

Proposition 5.5. Let $R \to R'$ be a flat morphism of $\mathbb{C}_p$–affinoid algebras. Then there is an isomorphism

$$\mathbf{D}^+_{\mathbf{dR},\sigma}(V_X) \otimes_R R' \xrightarrow{\sim} \mathbf{D}^+_{\mathbf{dR},\sigma}(V_X \otimes_R R')$$

Proof. By Theorem 4.3.7 of [Bel13], there is an isomorphism $\mathbf{D}^+_{\mathbf{dR}}(V_X) \otimes_R R' \xrightarrow{\sim} \mathbf{D}^+_{\mathbf{dR}}(V_X \otimes_R R')$. Since the isomorphism is $\overline{\mathbb{Q}}_p$–linear and $F_{\mathcal{P}}$–linear, it preserves the decomposition of each side as a $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$–module. In particular, it induces an isomorphism of the σ–parts of each side. □

5.2. **The finite slope subspace of $\mathcal{E}_D$ and applications.** In [Kis03], Kisin proved the existence of crystalline periods for finite slope overconvergent modular eigenforms by constructing the finite slope subspace of the eigencurve. His interpolation method has been generalised and modified by various people. In what follows we use Liu's generalisation [Liu12].

Definition 5.6. Let X be a rigid analytic space over $\mathbb{Q}_p$ and G be a topological group. A *family of p–adic representations V_X of G on X* is defined to be a locally free, coherent $\mathcal{O}_X$–module V_X, equipped with a continuous $\mathcal{O}_X$–linear G–action. We also denote the dual representation by V_X^*. If $\mathcal{U} \subset X$ is an admissible open subspace of X, then we write $V_{\mathcal{U}} := V_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{U}}$ for the restriction of the original family to $\mathcal{U}$. Further, if $x \in X$ is a point with residue field $k(x)$, then define $V_x := V_X \otimes_{\mathcal{O}_X} \mathrm{Spec}(k(x))$ to be the specialisation of V_X at x.

Definition 5.7. (Finite slope subspace) Consider a triple (X, α, V_X), where X is a reduced and separated rigid analytic space over $\mathbb{Q}_p$, V_X be a family of p-adic representations of $G_{F_{\mathcal{P}}}$ over X having 0 as a Hodge-Tate-Sen weight, and $\alpha \in \mathcal{O}(X)^\times$ a non vanishing global analytic function on X. Hence we can write the the Sen polynomial of V_X as $TQ(T)$, where $Q(T) \in \mathcal{O}(X)[T]$. We call an analytic subspace $X_{fs} \subseteq X$ a *finite slope subspace* of X with respect to the pair (α, V_X) if it satisfies the following conditions:

(1) For every integer $j \leqslant 0$, the subspace $(X_{fs})_{Q(j)}$ is scheme-theoretically dense in X_{fs}.
(2) For any affinoid algebra R over $\mathbb{Q}_p$ and any morphism $g : \mathrm{Sp}(R) \to X$ which factors through $X_{Q(j)}$ for all integers $j \leqslant 0$, the morphism g factors

through X_{fs} if and only if the natural map

$$\iota_n : \mathbf{D}^{\dagger}_{\mathbf{rig}}(V_R^*)^{\varphi^f=\alpha,\Gamma=1} \to \mathbf{D}^{+,\mathbf{n}}_{\mathbf{dif}}(V_R^*)^{\Gamma}$$

is an isomorphism for all n sufficiently large.

In [Liu12], Liu proves the following as Theorem 3.3.1:

Theorem 5.8. For every triple (X, α, V_X) as in Definition 5.7, X has a unique finite slope subspace X_{fs} associated to the pair (α, V_X).

Let $\iota : \tilde{\mathcal{E}}_D \to \mathcal{E}_D$ denote the normalisation of the eigencuve $\mathcal{E}_D$, $V_{\tilde{\mathcal{E}}_D}$ the universal family of Galois representations (of $G_{F_{\mathcal{P}}}$) on $\tilde{\mathcal{E}}_D$, and $\alpha_{\tilde{\mathcal{E}}_D}$ the pullback from $\mathcal{E}_D$ to $\tilde{\mathcal{E}}_D$ of the analytic function giving the $U_{\mathcal{P}}$-eigenvalue on $\mathcal{E}_D$.

Proposition 5.9. The finite slope subspace of $\tilde{\mathcal{E}}_D$ with respect to $(\alpha_{\tilde{\mathcal{E}}_D}, V_{\tilde{\mathcal{E}}_D})$ is $\tilde{\mathcal{E}}_D$.

Proof. Using [Liu12, Theorem 4.1.3], it is enough to show that the family of Galois representations $V_{\tilde{\mathcal{E}}_D}$ is a weakly refined family in the sense of Bellaiche and Chenevier (see Definition 0.3.1 in [Liu12]). This follows by considering Z (as in Liu's definition 0.3.1) to be the set of all points on $\mathcal{E}_D$ whose Galois representations are attached to classical eigenforms of level $H_1(N) \times K_1(1)$. Z being Zariski dense in $\mathcal{E}_D$ follows from the classicality criterion Theorem 2.24 (as explained in the proof of Proposition 3.6). That Z satisfies the rest of the desired conditions follows from well-known properties of modular Galois representations in level away from $\mathcal{P}$ (recounted, for example, in [Din14, Proposition 4.2.7]). □

Let $\mathcal{X}$ be a smooth rigid analytic space and $g : \mathcal{X} \to \mathcal{E}_D$ be a morphism. Then g lifts to a unique morphism $\tilde{g} : \mathcal{X} \to \tilde{\mathcal{E}}_D$. We denote by $V_{\mathcal{X}}$ the pullback of $V_{\tilde{\mathcal{E}}_D}$ via $\tilde{g}$, and by $\alpha_{\mathcal{X}}$ the pullback of $\alpha_{\tilde{\mathcal{E}}_D}$ to $\mathcal{X}$. In particular, if $\mathcal{U} \subset \mathcal{X}$ is an affinoid open, $(V_{\mathcal{U}}, \alpha_{\mathcal{U}})$ has an obvious sense. When there is no confusion, we will denote $\alpha_{\mathcal{X}}, \alpha_{\mathcal{U}}$ simply by α.

Corollary 5.10. Let $\mathcal{X}$ be as above, then the finite slope subspace of $\mathcal{X}$ with respect to $(\alpha_{\mathcal{X}}, V_{\mathcal{X}})$ is $\mathcal{X}$.

Proof. This is straightforward using Proposition 5.9. See Proposition 3.1.1 of [Liu12] for a proof. □

Proposition 5.11. Let $\mathcal{X}$ be as above and $\mathcal{U} = \mathrm{Sp}(R)$ an affinoid open subset. The natural map

$$\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{U}})^{\varphi^f=\alpha} \to \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{U}})$$

is an isomorphism and both sides are locally free R-modules of rank 1.

Proof. By Theorem 3.3.4 of [Liu12] and taking σ-components, the natural map

$$(\mathbf{D}^\dagger_{\mathbf{rig},\sigma}(V^*_{\mathcal{U}}))^{\varphi^f=\alpha,\Gamma=1} \to (\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}})/t^k)^\Gamma$$

is an isomorphism for $k > \log_{|\varpi|} |\alpha|_{\sup,\mathcal{U}}$. This implies that the induced natural map

$$(\mathbf{D}^\dagger_{\mathbf{rig},\sigma}(V^*_{\mathcal{U}}))^{\varphi^f=\alpha,\Gamma=1} \to \varprojlim_k (\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}})/t^k)^\Gamma \cong (\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}}))^\Gamma$$

is an isomorphism. Applying Proposition 5.4, it follows that the natural map

$$\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{U}})^{\varphi^f=\alpha} \to \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{U}})$$

is an isomorphism. To prove that $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{U}})^{\varphi^f=\alpha}$ is a locally free R-module of rank 1, we prove the same statement for $(\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}})/t^k)^\Gamma$. This module is finitely generated and torsion free over R. Since $\mathcal{U} = \mathrm{Sp}(R)$ is smooth, it follows that

$(\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}})/t^k)^\Gamma$ is a locally free R-module. To calculate its rank, we specialise to a point whose weight is not an integer. Let $x \in \mathcal{U}$ be a point of non-integral weight with residue field $k(x)$. By [Liu12, Corollary 1.5.6]

$$(\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}})/t^k)^\Gamma \otimes_R k(x) \cong (\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_x)/t^k)^\Gamma.$$

By [Liu12, Corollary 1.5.7] the right hand side has rank 1 over $k(x)$. This proves that $(\mathbf{D}^+_{\mathbf{dif},\sigma}(V^*_{\mathcal{U}})/t^k)^\Gamma$ is a locally free R-module of rank 1.

□

Let $\mathcal{D}$ be the closed unit disc over $\mathbb{Q}_p$. For non-negative integers $m > n \geqslant 0$, let $R_n := \mathbb{Q}_p\langle p^{-n}T\rangle$ denote the ring of convergent functions on the closed unit disk of radius p^{-n}, and set $R_{n,m} := \mathbb{Q}_p\langle p^{-n}T, p^m T^{-1}\rangle$ to be the ring of convergent functions on the annulus with outer radius p^{-n} and inner radius p^{-m}. Note that $\mathcal{D} = \mathrm{Spm}(R_0)$.

Finally, we record a lemma that will be used later.

Lemma 5.12. For any $x \in R_0\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^+_{\mathbf{dR}}$, if the image of x in $R_{0,1}\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^+_{\mathbf{dR}}$ lands inside $R_{0,1}\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^+_{\mathbf{cris}}$, then x must be an element of $R_0\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^+_{\mathbf{cris}}$.

Proof. This is Lemma 5.6 of [DL14]. □

6. The eigencurve $\mathcal{E}_D$ is proper

In this chapter, we will prove that the quaternionic eigencurve $\mathcal{E}_D$ is proper, in the sense that:

Theorem 6.1. Let $\mathcal{D}$ be the closed unit disk in $\mathbb{C}_p$, $\mathcal{D}^\times$ the punctured unit disk (i.e. $\mathcal{D}$ with the origin 0 removed), $\mathcal{E}_D$ the eigencurve, and $\mathcal{W}$ the weight space. Suppose that there is a rigid analytic morphism $h : \mathcal{D}^\times \to \mathcal{E}_D$ such that $\pi \circ h$ extends to a rigid analytic morphism $\mathcal{D} \to \mathcal{W}$. Then, there exists a unique morphism $\tilde{h} : \mathcal{D} \to \mathcal{E}_D$ which makes the following diagram commutative.

$$\begin{tikzcd}
\mathcal{D}^\times \arrow[r, "h"] \arrow[d, hook] & \mathcal{E}_D \arrow[d, "\pi"] \\
\mathcal{D} \arrow[r] \arrow[ur, dotted, "\exists \tilde{h}"] & \mathcal{W}
\end{tikzcd}$$

The theorem states that under the given assumptions, the map h extends to the origin, and so the eigencurve $\mathcal{E}_D$ satisfies a form of the *valuative criterion of properness.* We follow the method of [DL14]. First we note that it is enough to prove the above result for the normalisation $\tilde{\mathcal{E}}_D$ since any map h as above factors uniquely through the natural map $\tilde{\mathcal{E}}_D \to \mathcal{E}_D$. Using Theorem 4.3, it is enough to prove the result with $\tilde{\mathcal{E}}_D$ replaced with $\tilde{\mathcal{C}}_D^{red}$, the normalisation of $\mathcal{C}_D^{red}$. By construction, there is a family of pseudo-representations on $\mathcal{C}_D^{red}$. Pulling back this family via the natural morphism $\tilde{\mathcal{C}}_D \to \mathcal{C}_D$, we obtain a family of pseudo-representations on $\tilde{\mathcal{C}}_D^{red}$. Since $\tilde{\mathcal{C}}_D^{red}$ is a smooth rigid analytic curve, by Theorem 5.1.2. of [CM98], this family of rank 2 pseudo-representations is indeed the pseudo-representation attached to a genuine family of Galois representations on $\tilde{\mathcal{C}}_D^{red}$. We denote the restriction of this family to $G_{F_\mathcal{P}}$ by V, and we write $V_{\mathcal{D}^\times}$ for the pullback of V to $\mathcal{D}^\times$ (note that h factors through $\tilde{\mathcal{C}}_D^{red}$). The strategy of the proof is to extend $V_{\mathcal{D}^\times}$ to the origin, and to prove that the eigenform (corresponding to the Galois representation at the origin) has finite slope. The finite slope property turns out to be the hardest thing to prove, and requires looking at the crystalline periods.

It is enough to prove the theorem for each component $\tilde{\mathcal{C}}^{red}_{D,\bar{\rho}}$ corresponding to a choice of modular mod p Galois representation $\bar{\rho}$ of tame level N.

By construction of $\mathcal{C}_D$, there are projections $\mathcal{C}_D^{red} \to X_{\bar{\rho}}$, $\mathcal{C}_D^{red} \to \mathbb{G}_m$ and $\mathcal{C}_D^{red} \to \mathbb{A}^1$. By composing each projection with h, we obtain maps

$$pr_1 : \mathcal{D}^\times \to X_{\bar{\rho}}$$

$$pr_2 : \mathcal{D}^\times \to \mathbb{G}_m$$

$$\alpha_{\mathfrak{L}} : \mathcal{D}^\times \to \mathbb{A}^1 \quad (\mathfrak{L}|N)$$

If we can show that each projection extends to a map on $\mathcal{D}$, then we'll be done; this is because we will have a map $\tilde{h} : \mathcal{D} \to X_{\bar{\rho}} \times \prod_{\mathfrak{L}|N} \mathbb{A}^1 \times \mathbb{G}_m$. Since $h(\mathcal{D}^\times) \subset \mathcal{C}_D^{red}$ and $\mathcal{C}_D^{red}$ is closed in $X_{\bar{\rho}} \times \prod_{\mathfrak{L}|N} \mathbb{A}^1 \times \mathbb{G}_m$, by analyticity $\tilde{h}(\mathcal{D}) \subset \mathcal{C}_D^{red}$ too. Therefore, $\tilde{h}$ is the desired extension of h.

6.0.1. *Extending $pr_1 : \mathcal{D}^\times \to X_p$ and $\alpha_{\mathfrak{L}} : \mathcal{D}^\times \to \mathbb{A}^1$.* To prove that the maps pr_1 and the $\alpha_{\mathfrak{L}}$ extend to all of $\mathcal{D}$, we will utilise the following rigid analytic lemma as well as the proposition following it. The lemma will also be used in the proof that pr_2 extends to all of $\mathcal{D}$.

Lemma 6.2. Let $G \in \mathcal{O}(\mathcal{D}^\times)$. If there is a positive constant β such that $|G(x)| \leqslant \beta$ for all $x \in \mathcal{D}^\times$, then G extends uniquely to an element of $\mathcal{O}(\mathcal{D})$.

Proof. By rescaling G, we can assume without loss of generality that $|G(x)| \leqslant 1$ for all $x \in \mathcal{D}^\times$. Let $\mathcal{D}_{0,m} := \mathrm{Sp}(R_{0,m})$ be the closed annulus with outer radius

1 and inner radius p^{-m}. Since $|G| \leqslant 1$ on $\mathcal{D}_{0,m}$ for all $m \in \mathbb{N}$, we also have $G \in \mathcal{O}_{\mathbb{C}_p}\langle T, p^m T^{-1}\rangle$ for all $m \in \mathbb{N}$, and hence

$$G \in \bigcap_{m \geqslant 1} \mathcal{O}_{\mathbb{C}_p}\langle T, p^m T^{-1}\rangle = \mathcal{O}_{\mathbb{C}_p}\langle T\rangle$$

which tells us that $G \in \mathcal{O}(D)$. $\square$

Choose one of the maps $\alpha_{\mathfrak{L}}$. From the construction of $\mathcal{C}$, the projection $\alpha_{\mathfrak{L}}$ is given by a function $\mathcal{D}^\times \to \mathbb{A}^1$ that encodes the eigenvalues of the Hecke operator $T_{\mathfrak{L}}$, for some prime $\mathfrak{L}$ dividing N. However, since Hecke eigenvalues are always algebraic integers, $|\alpha_{\mathfrak{L}}(x)| \leqslant 1$ for all $x \in \mathcal{D}^\times$. Therefore, we can apply the lemma so that each $\alpha_{\mathfrak{L}}$ extends to a function on $\mathcal{D}$.

It remains to show that pr_1 extends to all of D. Recall that X_p can be viewed as the disjoint union of some $X_{\overline{V}}$. Since $D^\times$ is connected, $h(D^\times)$ lands in exactly one of these $X_{\overline{V}}$. Therefore, we have:

Proposition 6.3. The morphism $pr_1 : \mathcal{D}^\times \to X_{\bar{\rho}}$ extends to all of $\mathcal{D}$.

Proof. We know that $pr_1 : \mathcal{D}^\times \to X_{\bar{\rho}}$, where $X_{\bar{\rho}}$ is the generic fibre of $\mathrm{Spf}(R_{\bar{\rho}})$. Therefore, for any $x \in \mathcal{D}^\times$ and $f \in R_{\bar{\rho}}$, one has $|pr_1^*(f)(x)| = |f(pr_1(x))|$, which is $\leqslant 1$. By the previous lemma, this implies $pr_1^*(f)$ can be extended to an element of $\mathcal{O}_{\mathbb{C}_p}\langle T\rangle$ for any $f \in R_{\bar{\rho}}$. This corresponds to a homomorphism of rings $R_{\bar{\rho}} \to \mathcal{O}_{\mathbb{C}_p}\langle T\rangle$ which gives the extension $\tilde{pr}_1 : \mathcal{D} \to X_{\bar{\rho}}$. $\square$

6.0.2. *Extending* $pr_2 : D^\times \to \mathbb{G}_m$. The map pr_2 encodes a system of $U_{\mathcal{P}}$–eigenvalues parametrised by $D^\times$. Knowing that these are algebraic integers and using the previous lemma lets us extend pr_2 to a map $\tilde{pr}_2 : D \to \mathbb{A}^1$. The last, and most difficult, thing to do is to show that $\tilde{pr}_2(0) \neq 0$, which will imply $\tilde{pr}_2 : D \to \mathbb{G}_m$.

Let $\alpha_{\mathcal{C}_D}$ be the analytic function on $\mathcal{C}_D$ giving the $U_{\mathcal{P}}$-eigenvalue. We define $\alpha = h^*(\alpha_{\mathcal{C}_D})$. It is clear from definition of $\mathcal{C}_D$ that α is nothing but the map $pr_2 : \mathcal{D}^\times \to \mathbb{G}_m$. By the previous proposition, α extends to an element of $\mathcal{O}(D)^\times$ which we still denote by α.

We start by pulling back the universal pseudo-representation on $X_{\bar{\rho}}$ along the map $\tilde{pr}_1 : \mathcal{D} \to X_{\bar{\rho}}$ to get a family of pseudo-representations on $\mathcal{D}$. Since $\mathcal{D}$ is smooth, this family comes from a family of $G_{F_{\mathcal{P}}}$-representations $V_{\mathcal{D}}$ on $\mathcal{D}$.

Proposition 6.4. $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{D}}) = \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$.

Proof. Since $\mathbf{B}^+_{\mathbf{cris}} \subset \mathbf{B}^+_{\mathbf{dR}}$, we certainly have $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{D}}) \subset \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$. For the reverse inclusion, let $x \in \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$. Let $U = \mathrm{Sp}(R_{0,1})$ as in Lemma 5.12. Applying Proposition 5.11, we deduce that

$$\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_U)^{\varphi^f=\alpha} \xrightarrow{\sim} \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_U)$$

and, in particular, $\mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_U) \subset \mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_U)$. Therefore, we can apply Lemma 5.12 which tells us that $x \in \mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{D}})$. □

6.0.3. *The proof that $\tilde{pr}_2 : D \to \mathbb{A}^1$ satisfies $\tilde{pr}_2(0) \neq 0$.* We have our family $G_{F_{\mathcal{P}}}$–representations $V_{\mathcal{D}}$, and its dual $V^*_{\mathcal{D}}$. Recall our earlier notation: for any affinoid $\mathcal{U} \subset \mathcal{D}$, let $V_{\mathcal{U}} := V_{\mathcal{D}} \otimes_{\mathcal{O}_D} \mathcal{O}_{\mathcal{U}}$ denote the restriction of the family to $\mathcal{U}$. Also, let V_0 denote the restriction of the family to the origin. The aim is to specialise $V^*_{\mathcal{D}}$ to the origin, and prove that $\alpha(0)$ is a nonzero eigenvector for Frobenius in $\mathbf{D}^+_{\mathbf{cris},\sigma}$ of the specialisation V_0. This will end the proof because we already know that the Frobenius map is injective, and $\tilde{pr}_2(0) = \alpha(0) \neq 0$.

Recall that for $m > n \geqslant 0$ we defined $R_n := \mathbb{Q}_p\langle p^{-n}T\rangle$ the ring of convergent functions on the closed unit disk of radius p^{-n} and $R_{n,m} = \mathbb{Q}_p\langle p^{-n}T, p^mT^{-1}\rangle$ the ring of convergent functions on the closed annulus of inner radius p^{-m} and outer radius p^{-n}. Let $U_m = \mathrm{Sp}(R_{0,m})$. By Proposition 5.5 the formation of $\mathbf{D}^+_{\mathbf{dR},\sigma}$ commutes with flat base change, Hence, there is an isomorphism

$$\mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})\widehat{\otimes}_{R_0}R_{0,m} \xrightarrow{\sim} \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{U_m}).$$

If $a \in \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$, denote the image of $a \otimes 1$ under the above isomorphism by $a|_{U_m}$. Proposition 5.11 applied with $\mathcal{X} = \mathcal{D}^\times$ and $\mathcal{U} = U_m$ informs us that the right hand side is a locally free $R_{0,m}$–module of rank 1, so in particular $\mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}}) \neq 0$. We can go ahead and choose an element $a' \in \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$ for which $a'|_{U_m} \neq 0$. By dividing by a power of T, we can further assume $a'(0) \neq 0$. The specialisation of a' to the origin is our candidate for the non-zero eigenvector of Frobenius.

Returning to the global picture, note that Theorem 6.4 says $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{D}}) = \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$, so we can regard a as an element of $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{D}})$ too, and by restricting to the annulus U, we have $a|_{U_m} \in \mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{U_m})$. On the other hand, Proposition 5.11 tells us there is an isomorphism of locally free $R_{0,m}$–modules of rank 1:

$$\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{U_m})^{\varphi^f=\alpha} \xrightarrow{\sim} \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{U_m})$$

Viewing $a \in \mathbf{D}^+_{\mathbf{dR},\sigma}(V^*_{\mathcal{D}})$, its restriction $a|_{U_m}$ is in the right–hand side, so the isomorphism says $a|_{U_m} \in \mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{U_m})^{\varphi^f=\alpha}$ too. Therefore we have found that

Lemma 6.5. The image of the restriction map $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{\mathcal{D}}) \to \mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{U_m})$ lands in $\mathbf{D}^+_{\mathbf{cris},\sigma}(V^*_{U_m})^{\varphi^f=\alpha}$.

We see that the f–th power of Frobenius acts on $a|_{U_m}$ via α, i.e.:

$$\varphi^f(a|_{U_m}) = \alpha|_{U_m}a|_{U_m}$$

or equivalently

$$(\varphi^f - \alpha)|_{U_m}(a|_{U_m}) = 0$$

Note the above is true for any $m \in \mathbb{N}$. Since the Zariski closure of the annuli $\bigcup_{m \geqslant 1} U_m$ is all of $\mathcal{D}$, this implies that we must have $\varphi^f(a) = \alpha a$ on all of $\mathcal{D}$. Specialising to the origin, we have that $\varphi^f(a_0) = \alpha(0)a_0$ in $\mathbf{D}^+_{\mathbf{cris},\sigma}(V_0^*)$, and by injectivity of Frobenius, we deduce $\alpha(0) \neq 0$.

This completes the proof.

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